

The three-nucleon parameter

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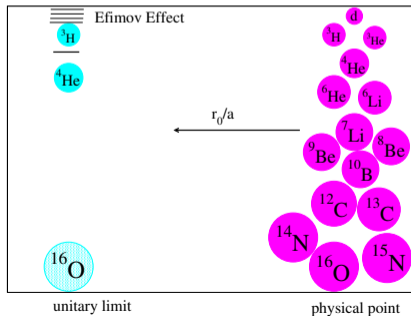
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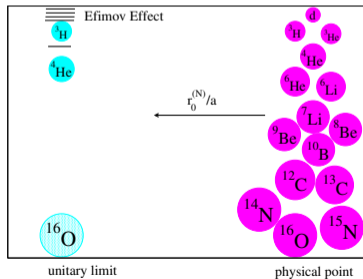


The trip

- ▶ The trip follows a particular path, it is determined by the ratio r_0/a
- ▶ a is the scattering length, it characterizes each point of the path
- ▶ r_0 is a characteristic length, constant along the path. It is different for each nucleus, $r_0^{(N)}$. Its determination is part of this talk.
- ▶ The trip consists in moving nuclei from their physical point to the unitary limit

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The Starting point: The nuclear physical point

- ▶ The nuclear physical point is characterized by the nuclear potential

$$V = \sum_{i < j} V_{2N}(i, j) + \sum_{i < j < k} V_{3N}(i, j, k) + \dots$$

- ▶ The modern theory of the nuclear interaction is based on effective field theory (EFT) framework preserving chiral symmetry, chiral-EFT. Potentials derived from it are called chiral potentials.
- ▶ The potential depend on a certain number of constants, the low energy constants (LECs), fixing the strength of the different terms in the chiral perturbation series.

$$V_{2N}(1, 2) = \sum_n v_n([C_i, D_i], r_{12}) \mathcal{O}_n(1, 2)$$

- ▶ The physical point is a point in the phase space of these parameters

The Starting point: The nuclear physical point

- ▶ The LECs, around 30, are determined from a fit to the world NN data, more than 5000 data points.
- ▶ At leading order (LO) the $2N$ potential has two LECs

$$V_{2N}^{LO}(1, 2) = C_0 e^{-(\Lambda r_{12}/4)^2} \mathcal{P}_{01} + C_1 e^{-(\Lambda r_{12}/4)^2} \mathcal{P}_{10} + V_{OPEP}(1, 2)$$

\mathcal{P}_{ST} is a projector on the spin-isospin channel ST and Λ is the cutoff.

- ▶ The two LECs, C_0 and C_1 , are determined by the singlet and triplet np scattering lengths:
 $a_0 = -23.74 \pm 0.02$ fm and $a_1 = 5.419 \pm 0.007$ fm
- ▶ The physical point is characterized by these two values

The Arrival: The unitary point

- ▶ The unitary point is characterized by the infinite value of the scattering lengths:
 $a_0, a_1 \rightarrow \pm\infty$
- ▶ We need to create a path from the physical point to the unitary point. How to do that?

The Arrival: The unitary point

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- ▶ We need to create a path from the physical point to the unitary point. How to do that?

Low energy properties of the np system:

- ▶ We observe that the scattering lengths are large with respect to the effective range
 $a_0 = -23.74 \pm 0.02 \text{ fm}$ $a_1 = 5.419 \pm 0.007 \text{ fm}$
 $r_e^0 = 2.77 \pm 0.05 \text{ fm}$ $r_e^1 = 1.753 \pm 0.008 \text{ fm}$
- ▶ The np system in $S=0$ has a virtual state close to threshold, $E_v = -0.066 \text{ MeV}$
In $S=1$, the deuteron is a very loosely bound state, $E_d = -2.224 \text{ MeV}$

Low-energy quantities

Solving the Schrödinger equation, $H\Psi = E\Psi$, we consider three regions:

- ▶ The discrete region $E < 0$. The wave function $\Psi(r \rightarrow \infty) \rightarrow C_a e^{-k_a r}/r$
- ▶ The zero-energy region $E = 0$. The wave function $\Psi(r \rightarrow \infty) \rightarrow 1 - a/r$
- ▶ The continuum region $E > 0$. The w.f. $\Psi(r \rightarrow \infty) \rightarrow F_L(kr) + \tan \delta_L G_L(kr)$

C_a is the asymptotic normalization constant, a is the scattering length and δ is the phase-shift. Close to the $E = 0$ threshold these quantities are related.

In particular the s -wave phase-shift at low energies ($E = \hbar^2 k^2/m$) results

$$k \cot \delta = -\frac{1}{a} + \frac{1}{2} r_e k^2 + \dots$$

This is the effective range expansion and r_e is the effective range

Another important relation is: the binding energy $E(a \rightarrow \pm\infty) \rightarrow -\hbar^2/ma^2$

Shallow states: correlations between low-energy quantities

- ▶ A bound or virtual state is a pole of the S -matrix

$$S(k) = e^{2i\delta} = \frac{e^{i\delta}}{e^{-i\delta}} = \frac{\cos \delta + i \sin \delta}{\cos \delta - i \sin \delta} = \frac{k \cot \delta + ik}{k \cot \delta - ik}$$

the energy of the pole is given by the equation $k \cot \delta = ik$

- ▶ On the other hand the second order effective range expansion is

$$k \cot \delta = -\frac{1}{a} + \frac{1}{2}r_e k^2 + \dots$$

- ▶ If the second order expansion of $k \cot \delta$ is sufficient for determining the energy of the pole, the state is shallow!

Making $k = i\kappa$ the pole equation is obtained

$$\kappa = \frac{1}{a} + \frac{1}{2}r_e \kappa^2 + \dots$$

The deuteron is shallow

$$\kappa = \frac{1}{a} + \frac{1}{2}r_e\kappa^2 + \dots$$

To verify if the deuteron is shallow we use experimental data:

$$E_d = -2.2245 \text{ MeV} = -\hbar^2\kappa^2/m \rightarrow \kappa = 0.2316 \text{ fm}^{-1}$$

Replacing this value in the second term and using the experimental values of a_1, r_e^1 we obtain:

$$\frac{1}{a_1} + \frac{1}{2}r_e^1\kappa^2 = 0.2315 \text{ fm}^{-1}$$

The deuteron is a shallow state!

The S -matrix for shallow states

The simplest S -matrix representing a system with a shallow bound or virtual state is the following. It is equivalent to the second order effective range expansion:

$$S(k) = \frac{k + i\kappa}{k - i\kappa} \frac{k + i\kappa_B}{k - i\kappa_B}$$

with κ, κ_B the two solutions of the second order pole equation:

$$\kappa = \frac{1}{r_e} \left[1 - \sqrt{1 - 2r_e/a} \right] \quad \rightarrow r_e/a < 0.5$$

$$\kappa_B = \frac{1}{r_e} \left[1 + \sqrt{1 - 2r_e/a} \right]$$

When the energy pole (or binding momentum), $\kappa = 1/a_B > 0$, it represents a bound state. When $\kappa = 1/a_B < 0$ it represents a virtual state.

The second pole, sometimes called redundant pole, $\kappa_B = 1/r_B > 0$, is always positive. Moreover $r_B = a - a_B$.

The two-pole S -matrix at the physical point

- ▶ The path from the physical point to the unitary point is characterized by the two-pole S -matrix representing one shallow state, virtual or bound

$$S(k) = \frac{k + i/a_B}{k - i/a_B} \frac{k + i/r_B}{k - i/r_B}$$

- ▶ The energy pole is described by the energy length a_B

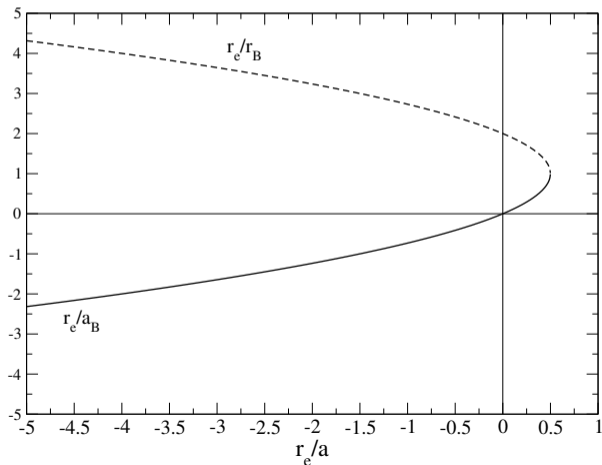
$$1/\kappa = a_B \longrightarrow E_2 = -\hbar^2 \kappa^2 / m = -\hbar^2 / m a_B^2$$

- ▶ E_2 is a bound or virtual state when $a_B > 0$ or $a_B < 0$
- ▶ the second pole is described by the length $r_B = a - a_B$

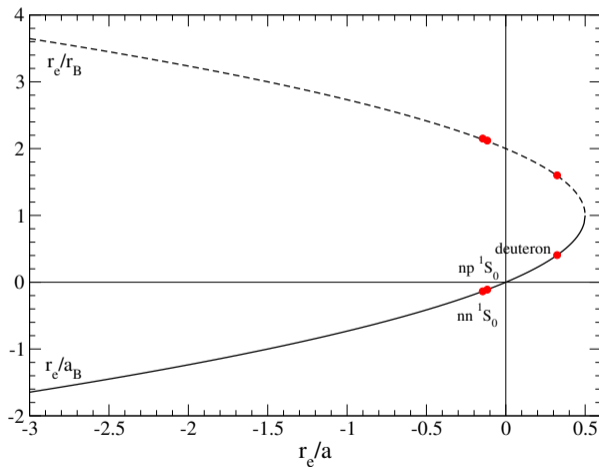
- ▶ At the physical point: $S = 1 \left\{ \begin{array}{l} a_B = 4.32 \text{ fm} \\ r_B = 1.10 \text{ fm} \end{array} \right.$, $S = 0 \left\{ \begin{array}{l} a_B = -25.05 \text{ fm} \\ r_B = 1.31 \text{ fm} \end{array} \right.$

The two-pole S -matrix form the universal window

$$\frac{r_e}{a_B} = 1 - \sqrt{1 - 2r_e/a}, \quad \frac{r_e}{r_B} = 1 + \sqrt{1 - 2r_e/a}$$

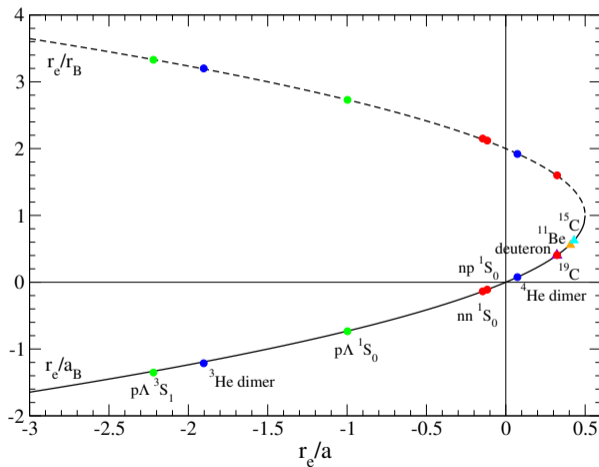


The two-nucleon system inside the universal window



Systems are put on the figure by their experimental data

Physical systems inside the universal window



Systems are put on the figure by their experimental data

Effective description (scale invariance)

- ▶ The S-matrix

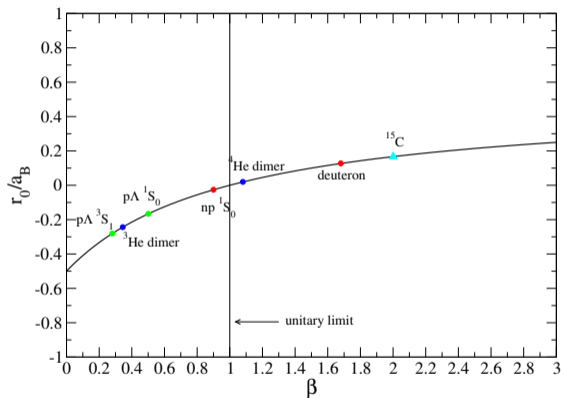
$$S(k) = \frac{k + i/a_B}{k - i/a_B} \frac{k + i/r_B}{k - i/r_B}$$

is exactly represented by the Eckart potential:

$$V(r) = -2 \frac{\hbar^2}{m r_0^2} \frac{\beta e^{-r/r_0}}{(1 + \beta e^{-r/r_0})^2}$$

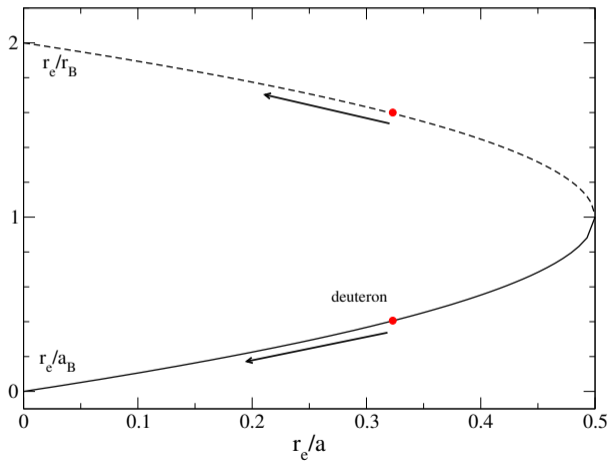
$$\left\{ \begin{array}{l} a = 4r_0 \frac{\beta}{\beta-1} \\ a_B = 2r_0 \frac{\beta+1}{\beta-1} \end{array} \right. \quad \left\{ \begin{array}{l} r_e = 2r_0 \frac{\beta+1}{\beta} \\ r_B = 2r_0 \rightarrow \text{the second pole!} \end{array} \right.$$

The universal window in terms of the Eckart parameters



$$V(r) = -2 \frac{\hbar^2}{mr_0^2} \frac{\beta e^{-r/r_0}}{(1 + \beta e^{-r/r_0})^2}$$

Our trip: $r_0^{(2)} \equiv \text{constant} \implies r_B \equiv \text{constant}$



Motivation of the trip

- ▶ The nuclear system, as well many other systems, are inside the universal window
- ▶ The universal window is characterized by scale invariance
- ▶ Scale invariance is not a property of the underlying theory but is an emergent invariance for particular values of low-energy quantities and, accordingly, for particular values of the interaction parameters

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- ▶ We can consider the following two main ingredients
 - ▶ The microscopic theory for the nuclear interaction
 - ▶ The scale invariance

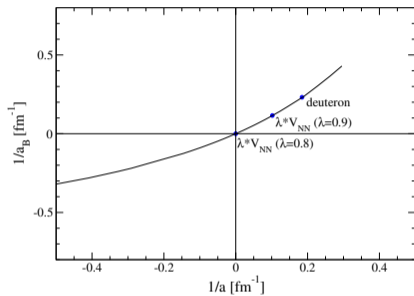
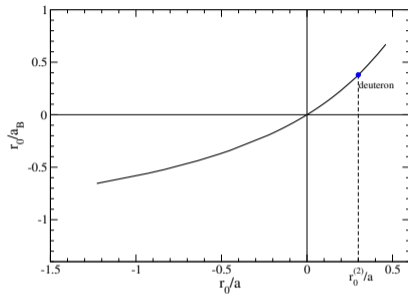
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- ▶ We can consider the following two main ingredients
 - ▶ The microscopic theory for the nuclear interaction
 - ▶ The scale invariance

- ▶ The trip will help to see how scale invariance manifests and, hopefully, how to incorporate it in the effective description of nuclei.

Starting the trip: the deuteron trip



→

$$r_0^{(2)} = 1.56 \text{ fm}$$

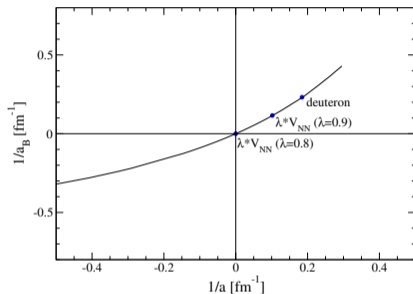
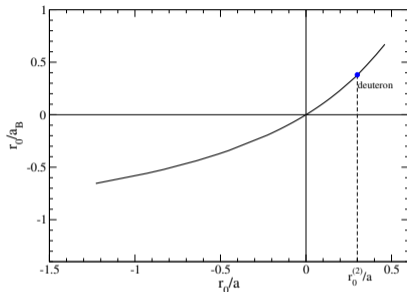
assigning dimensions

Scaling np NN potential

in $S, T = 1, 0$

$$V_{NN} \rightarrow \lambda V_{NN}$$

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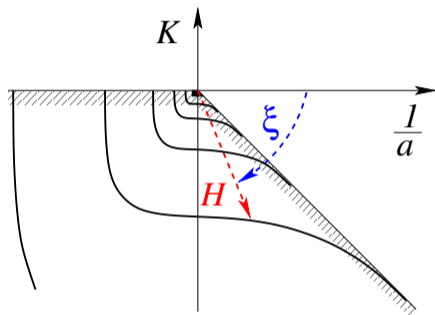
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The np data, represented by NN interaction has been moved from the physical point to the unitary point

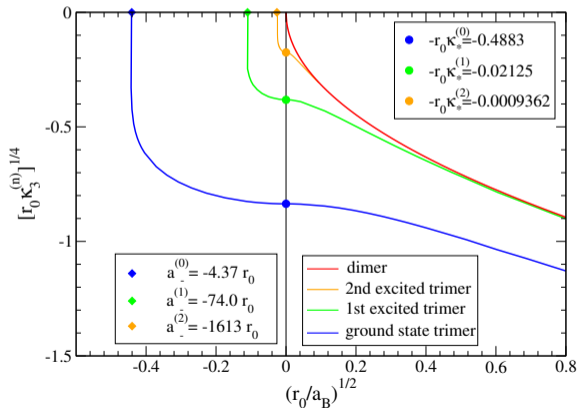
The three-body system inside the universal window



The Efimov plot: The three-body sector is scale invariant, K_* , the three-body parameter, is the binding momentum at the unitary limit. It fixes the branch in which the system is located

The three-body universal window using a gaussian characterization

The case of three bosons: $V = \sum_{ij} V_0 e^{-(r_{ij}/r_0)^2}$

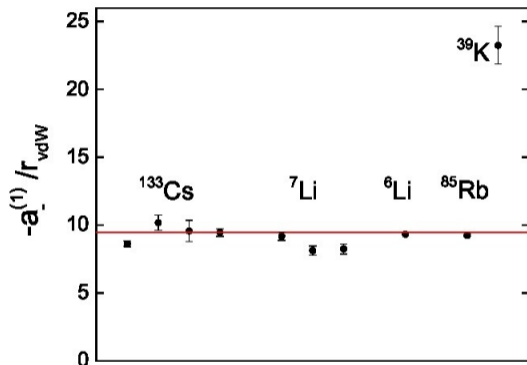


$$a_-^{(0)} \kappa_*^{(0)} = -2.14$$

$$a_-^{(1)} \kappa_*^{(1)} = -1.57$$

$$a_-^{(2)} \kappa_*^{(2)} = -1.51$$

van der Waals universality

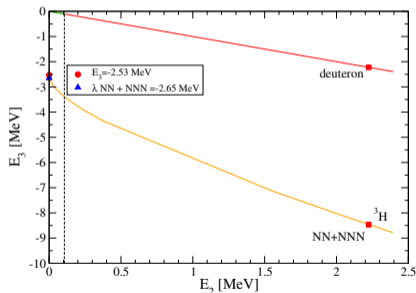
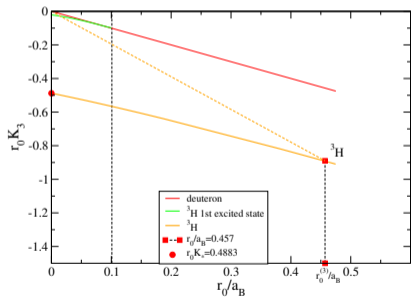


$$\begin{cases} a_{-}^{(0)} = (-9.69 \pm 0.20)r_{vdW} \\ \kappa_{*}^{(0)} = (0.226 \pm 0.004)/r_{vdW} \end{cases} \rightarrow a_{-}^{(0)} \kappa_{*}^{(0)} \approx -2.2$$

The three-body scale $r_0^{(3)}$ using the gaussian characterization

$$V(1,2,3) = \sum_{i<j} V(i,j) = \sum_{i<j} \left(V_0 e^{-(r/r_0)^2} \mathcal{P}_{01} + V_1 e^{-(r/r_0)^2} \mathcal{P}_{10} \right)$$

To construct the plot we vary the strength maintaining the gaussian range constant



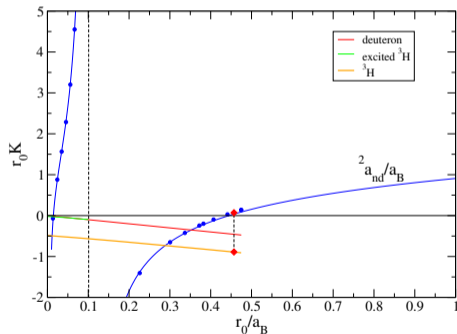
$$r_0^{(3)} = 1.98 \text{ fm}$$

assigning dimensions $\lambda \text{ NN} + \text{NNN}$

The three-nucleon system: correlations

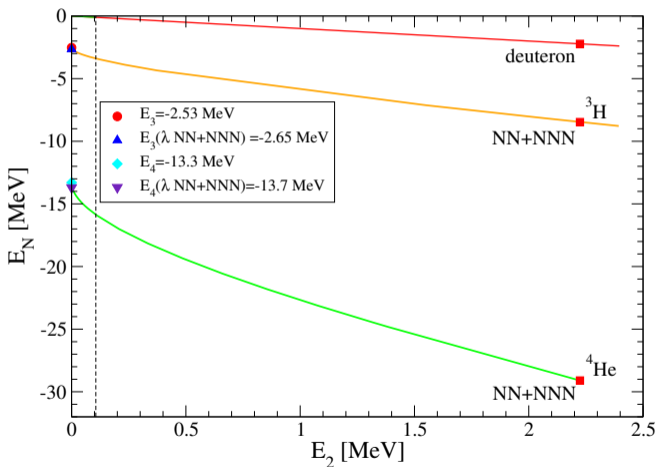
$$V(1,2,3) = \sum_{i<j} V(i,j) = \sum_{i<j} \left(V_0 e^{-(r/r_0)^2} \mathcal{P}_{01} + V_1 e^{-(r/r_0)^2} \mathcal{P}_{10} \right)$$

The neutron-deuteron scattering length: $a_{nd} = 0.645 \pm 0.010$ fm

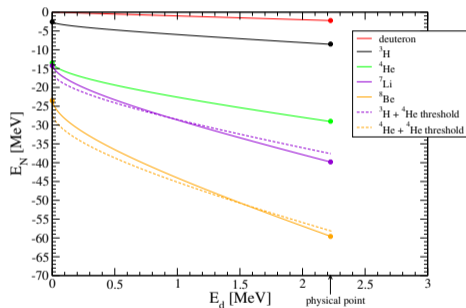
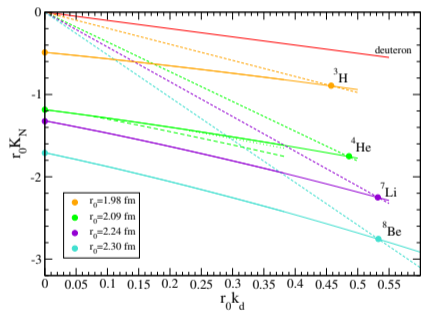


— $a_{nd}/a_B = d_1 + d_2 \tan[s_0 \ln(K_* r_0 (a_B/r_0) + \Gamma_3) + d_3]$

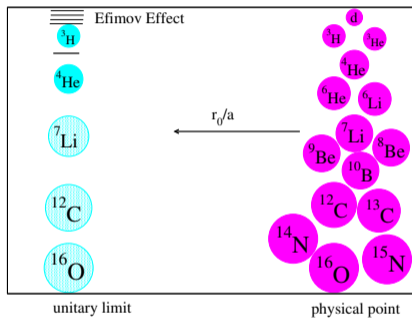
The three- and four-body scales, $r_0^{(3)}$ and $r_0^{(4)}$



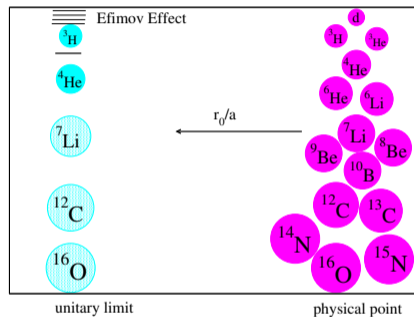
The N -body scales, $r_0^{(N)}$, for $A \leq 8$



The trip



The trip



$$V_N = \sum_{i < j} V(i, j, r_0^{(N)}, \beta_0) \mathcal{P}_{01} + \sum_{i < j} V(i, j, r_1^{(N)}, \beta_1) \mathcal{P}_{10}$$

Unifying the scales: the LO effective potential

- ▶ The scale invariance is encoded in the two-pole S -matrix
- ▶ The trip to the unitary point has shown the importance of two nuclear structures, the three-nucleon and four-nucleon systems. They form the thresholds from which the other nuclei emerge

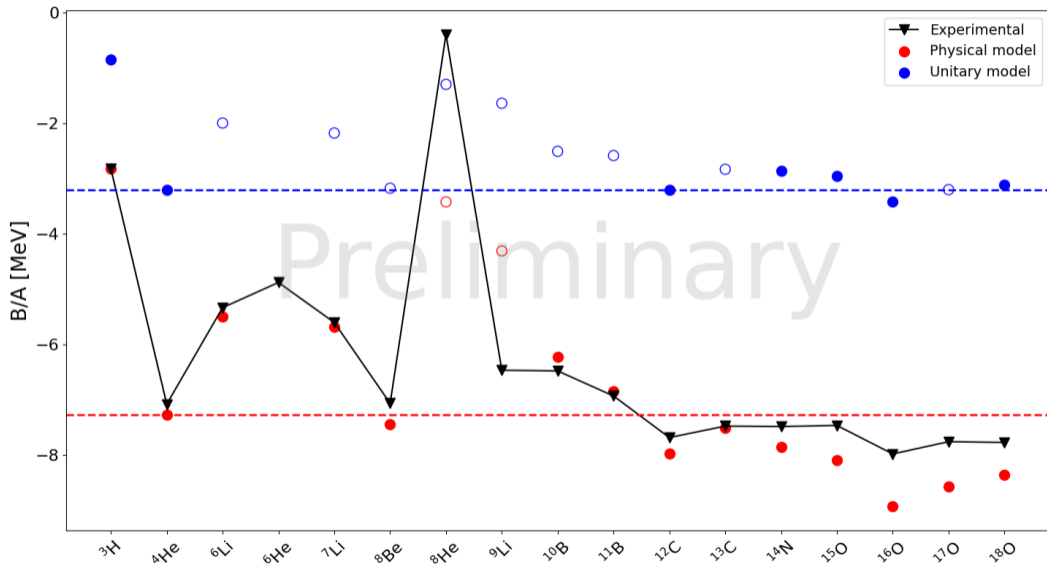
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- ▶ Accordingly we can use the following potential to describe nuclei ground state energies

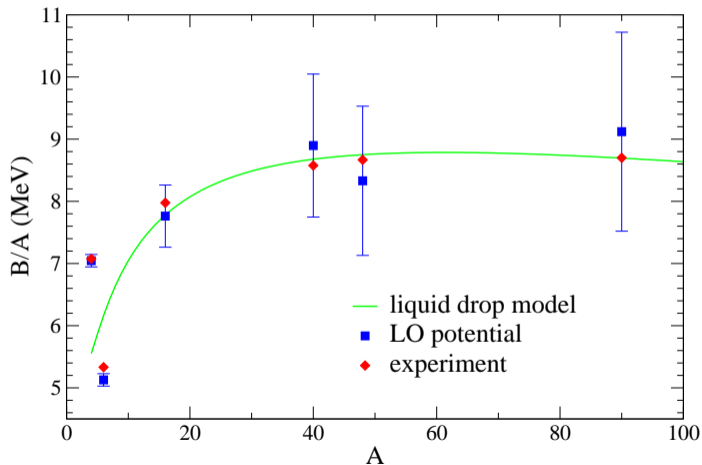
$$V_N = \sum_{i < j} V(i, j, r_0^{(2)}, \beta_0) \mathcal{P}_{01} + \sum_{i < j} V(i, j, r_1^{(2)}, \beta_1) \mathcal{P}_{10} + \sum_{i < j < k} W(i, j, k, r_3, \beta_3)$$

with β_3, r_3 fixed to reproduce $E(^3\text{H})$ and $E(^4\text{He})$

Mirroring the nuclear chart at the unitary limit



The LO potential at the physical point



Conclusions

- ▶ The nuclear system is well inside the universal window, accordingly it shows scale invariance
- ▶ Scale invariance manifests in particular correlations not well explained otherwise
- ▶ This is an emergent property, independent of the microscopic theory as many different systems are located inside this window
- ▶ It will be important to incorporate this invariance in the Ab Initio description of the nuclear structure
- ▶ From our trip we have seen the important structures at the unitary limit encoded in the three- and four-nucleon parameters

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I hope you have enjoyed the trip!