# Exclusive processes at B-factories in framework of Bethe-Salpeter equation

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A.N.Mitra memorial meeting, April 14, 2025

# Outline of Talk

- Introduction
- Bethe-Salpeter equation
- Charmonium Production processes:

  (a) $e^- + e^+ \longrightarrow \gamma^* \longrightarrow \gamma + \mathrm{H}$ ;

  (b) $e^- + e^+ \rightarrow \gamma^* + \gamma^* \rightarrow J/\psi + J/\psi$;

  (c) $e^- + e^+ \rightarrow \gamma^* + \gamma^* \rightarrow \eta_c + \eta_c$
- Conclusion

## Introduction: Exclusive processes at B-factories

- Exclusive processes refer to scattering events where all final-state particles are observed and identified:

- $e^- + e^+ \longrightarrow \gamma* \longrightarrow \gamma + H \ (H = \chi_{cJ} \,; \eta_c)$;
- $e^- + e^+ \longrightarrow \gamma* \longrightarrow J/\psi + J/\psi$;
- $e^- + e^+ \longrightarrow \gamma* \longrightarrow \eta_c + \eta_c$;
- $e^- + e^+ \longrightarrow \gamma* \longrightarrow J/\psi + \eta_c$ ;

- one of the most intriguing subjects, and remains one of the most vibrant areas of research in quarkonium physics.
- Bridge between perturbative and non-perturbative QCD.
- Provide ideal testing ground for fundamental theories, and continue to refine our understanding of strong interactions.

- Measurement of cross sections at 10.6 GeV:

- $e^-e^+ \rightarrow \gamma^* \longrightarrow J/\Psi + \eta_c$ Belle: PRL89, 142001 (2002), PRD70, 071102 (2004)]
- $e^-e^+ \rightarrow \gamma^* \longrightarrow J/\Psi + \chi_{cJ}$ at 10.6GeV, BABAR: [PRD72, 031101 (2005)]
- Led to rapid progress in theoretical description of charmonium

- Large discrepancy between leading order NRQCD prediction [Braaten and Lee, PRD67, 054007 (2003), PRD 72 (2005)] and Belle / BaBar data.
  - Resolved only when radiative corrections + relativistic corrections taken simultaneously [Bodwin, Lee, Yu, PRD77 (2008)].
- Additional processes:
  - e- e+ $\rightarrow \gamma^* \rightarrow \gamma + \chi_{cJ}$; e- e+ $\rightarrow \gamma^* \rightarrow \gamma + \eta_c$ (at √s = 10.52, 10.58, 10.867GeV) [S. Jia et al., (Belle) PRD98, 092015 (2018)].
  - e- e+ $\rightarrow \gamma^* \rightarrow \gamma\, \chi_{cJ}$ (at √s= 4- 4.6 GeV)

    [Ablikin et al.(BESIII), PRD104, 092001 (2021)].

  - e−e+ →γ*γ*→ J/ψ + J/ψ,: But absence of clear signal in Belle's measurement [K. Abe, et al., PRD 70, 071102(2004)].
- Theoretical Models: NRQCD, Light-cone expansion, Vector dominance model, Bethe-Salpeter equation.
- NRQCD factorization:

$$\sigma(e^- e^+ \to H + X) = \sum_n C_n < O_n^H >$$

# Motivation of Bethe-Salpeter equation:

1. Dynamical equation-based approach: Incorporates relativistic effects non-perturbatively even at lowest order in QCD.

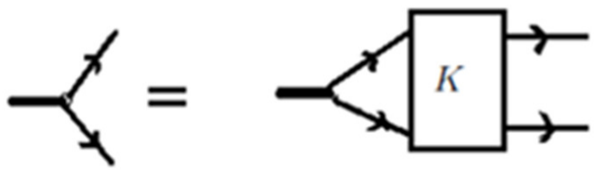
2. Relativistic effects of quark spins (incorporated through BS wave functions) + covariant description of constituent quarks within hadron.

3. BSE framework applicable to processes across different energy scales.

- Processes studied at Leading order:
  - $e^- + e^+ \rightarrow \gamma^* \rightarrow \gamma + H$ $(H = \chi_{co}$(nP), $\chi_{c1}$(nP), $\eta_c$(nS))
  - $e^- + e^+ \rightarrow \gamma^* + \gamma^* \rightarrow J/\psi + J/\psi$ ; $e^- + e^+ \rightarrow \gamma^* + \gamma^* \rightarrow \eta_c + \eta_c$ ;
- At large energy scale ($\sqrt{s}$= 10.6 GeV):
  - Lowest order QCD expansion in $\alpha_s$ is sufficient- the dominant contribution comes from lowest order diagrams in BSE.
  - Higher order $\alpha_s$ and $\alpha_{em}$corrections expected to be negligible, and hence ignored.

  $\Rightarrow$ Lowest order QCD diagrams dominate at $\sqrt{s}$~ 10.6 GeV

# Bethe-Salpeter equation




$$\Psi(P,q) = S_F(p_1) i \int \frac{d^4q'}{(2\pi)^4} K(q,q')\Psi(P,q')S_F(p_2)$$

- **BSE Overview:**
  - Describes relativistic bound states in QFT
  - Integral equation governing a **two-body system** with quarks.

- Reduction from 4D BSE → 3D form (under Covariant Instantaneous Ansatz)
- $K(q,q') = K(\hat{q},\hat{q}')$

- *Definitions:* $q = (\hat{q}_\mu, M\sigma)$

$\hat{q}_\mu = q_\mu - \frac{q.P}{P^2} P_\mu$ ; ($\hat{q}$.P=0)

$\sigma P_\mu = \frac{q.P}{P^2} P_\mu$ (Longitudinal to P),

4D volume element, $d^4q = d^3\hat{q} M d\sigma$

- $\hat{q}^2 = q^2 - \frac{(q.P)^2}{P^2} \geq 0$ (Lorentz-invariant)

A.N.Mitra, S.Bhatnagar, IJMPA07, 121 (1992);
G.L. Wang et el., JHEP 03, 209,(2016)
J.K. He, C.J. Fan, PRD 103 , 114006(2021)
G.L.Wang, et al.,PRD110, 113009 (2024);

- Properties of $\hat{q}^2$:
  - 1. Lorentz-invariant variable ⇛ Enhances applicability of BSE under CIA across wide range of energy scales.
  - 2. Ensures Lorentz-covariance of 3D forms of transition amplitudes + 3D Salpeter equations
  - **One-gluon-exchange like, vector-type kernel** allows to relate scalar part of confinement potential with the scalar part of the gluon propagator in the infrared domain.

- 3D Salpeter equations:

$$(M - \omega_1 - \omega_2)\psi^{++}(\hat{q}) = \Lambda_1^+(\hat{q})\Gamma(\hat{q})\Lambda_2^+(\hat{q})$$

$$(M + \omega_1 + \omega_2)\psi^{--}(\hat{q}) = -\Lambda_1^-(\hat{q})\Gamma(\hat{q})\Lambda_2^-(\hat{q})$$

$$\psi^{+-}(\hat{q}) = 0$$

$$\psi^{-+}(\hat{q}) = 0$$

$$\psi^{\pm\pm}(\hat{q}) = \Lambda_1^{\pm}(\hat{q})\frac{\not{P}}{M}\psi(\hat{q})\frac{\not{P}}{M}\Lambda_2^{\pm}(\hat{q})$$

$$\Lambda_i^{\pm} = \frac{1}{2\omega_i}\left[\frac{\not{P}}{M}\omega_i' \pm I(i)(im_i + \hat{q})\right]$$

Strategy for mass spectral calculations:

1. Start with the full 4D BS wave function:

$$\Psi^P(P,q) = \{\phi_1(P,q) + \not{P}\phi_2(P,q) + \not{q}\phi_3(P,q) + [\not{P},\not{q}]\phi_4(P,q)\}\gamma_5$$

$$\phi_j = \phi_j(q^2, q\cdot P, P^2)$$

CHL Smith, Ann. Phys. 53, 521 (1969);
R.Alkofer, L.W. Smekel, Phys. Rep. 353, 281(2001).

Earlier works: Had developed a Power counting rule to distinguish the leading Dirac structures with the sub-leading structures:
[S.Bhatnagar, S-Y.Li, JPG32, 949 (2006);
S.Bhatnagar, J. Mahecha, Y. Mangesha, PRD90,014034(2014)]

Under CIA:

$$\psi^P(\hat{q}) \approx \left[M\phi_1(\hat{q}) + \not{P}\phi_2(\hat{q}) + \not{\hat{q}}\phi_3(\hat{q}) + \frac{\not{P}\not{\hat{q}}}{M}\phi_4(\hat{q})\right]\gamma_5$$

$\Rightarrow$ $\phi_1$ ,$\phi_2$ (Leading amplitudes); $\phi_3$ ,$\phi_4$ (Sub–leading amplitudes)

$$\psi^{+-}(\hat{q}) = \psi^{-+}(\hat{q}) = 0$$

\- Equations of constraint- provide algebraic relationships (between $\phi_1$ and $\phi_3$); and (between $\phi_2$ and $\phi_4$)

$$(M-2\omega)\psi^{++}(\hat{q}) = -\Lambda_1^{+}(\hat{q})\Gamma(\hat{q})\Lambda_2^{+}(\hat{q})$$

$$(M+2\omega)\psi^{--}(\hat{q}) = \Lambda_1^{-}(\hat{q})\Gamma(\hat{q})\Lambda_2^{-}(\hat{q}),$$

$\Rightarrow$ *Coupled Salpeter equations*

3D Salpeter equations found to decouple in heavy-quark limit, to give Mass spectrum dependent on N + analytic forms of radial wave functions:

H.Negash, S.Bhatnagar, IJMPE25,1650059 (2016).

E.Gebrehana, S.Bhatnagar, H.Negash, PRD100, 054034 (2019)

S.Bhatnagar, L. Alemu, PRD97, 034021 (2018)

Radial wave functions employed for calculation of leptonic decays, Radiative E1, M1 transitions etc.:

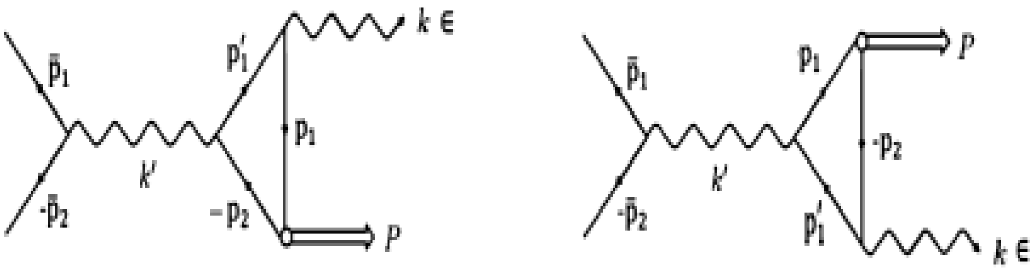
S.Bhatnagar, E.Gebrehana, PRDD102, 094024 (2020);

V.Guleria, E.Gebrehana, S.Bhatnagar, PRD 104, 094045(2021),

H.Negash, S.Bhatnagar, AHEP2017, 7306825 (2017),

## Cross section for $e^- + e^+ \to \gamma * \to \gamma\, X_{c0}$

Nucl. Phys. A1041, 122783 (2024)




- $M_{fi}=ie_Q^2[\overline{v_2}\gamma_\mu u_1]\frac{1}{s}\int \frac{d^4q}{(2\pi)^4} Tr[\overline{\Psi_s}(P,q)\,(\gamma.\epsilon')\,S_F(p_1')\gamma_\mu]$

Amplitude $M_{fi}$ expressed as: $M_{fi}= ie_Q^2[\overline{v_2}\gamma_\mu u_1]\, M_\mu$ ;

$M\mu = \, <\gamma, \chi_{c0}|\, J\mu\, |o>$ - Transition matrix element of e.m. current for $\gamma^* \to \gamma\, \chi_{c0}$

$$\psi(P,q) = f_1(q,P) - i\not{P} f_2(q,P) - i\not{q} f_3(q,P) - [\not{P},\not{q}] f_4(q,P),$$

[C. H. L. Smith, Ann. Phys. 53, 521 (1969)]

$$M^1_{fi} = -iee_Q^2[\bar{v}^{(s2)}(\bar{p}_2)\gamma_\mu u^{(s1)}(\bar{p}_1)]\frac{-1}{s}\times$$
$$\int \frac{d^3\hat{q}}{(2\pi)^3}\int \frac{M\,d\sigma}{2\pi i} Tr\left[\left(-\frac{\bar{\Psi}^{++}(\hat{q})(M-2\omega)}{\eta_3\eta_1} - \frac{\bar{\Psi}^{--}(\hat{q})(M+2\omega)}{\eta_4\eta_2}\right)\not{\epsilon}' S_F(p_1')\gamma_\mu\right],$$

Pole positions of quark-propagators $S_F(p_1)$, $S_F(-p_2)$ and $S_F(p'_1)$ in complex $\sigma$-plane:

$$\sigma_1^\pm = -\frac{1}{2} \mp \frac{\omega}{M} \pm i\epsilon \qquad \sigma_2^\pm = \frac{1}{2} \mp \frac{\omega}{M} \pm i\epsilon \qquad \beta^\pm = (-\frac{1}{2} - \frac{2E^2}{M^2}) \mp \frac{1}{M}\sqrt{\omega^2 + \frac{4E^4}{M^2}} \pm i\epsilon$$

$M_{fi} = i\,[\overline{v_2}\gamma_\mu u_1]\,M_\mu$ ;  $M_\mu = \beta_1 \epsilon'_\mu + \beta_2 (I.\epsilon') P_\mu + \beta_3 (I.\epsilon') k_\mu$

e.m. gauge invariance demands: $k_\mu M_\mu = 0 \;\Rightarrow\; \beta_2 = 0$

$$M_{fi} = [\bar{v}^{s_2}(p_2)\gamma_\mu u^{s_1}(p_1)][\beta_1 \epsilon'_\mu + \beta_3 (I.\epsilon') k_\mu]$$

(Lorentz-covariance of 3D amplitude under Covariant instantaneous ansatz)

e.m. form factors:

$$\beta_1 = \frac{8 e e_Q^2 N_S}{M^4 s} \int \frac{d^3\hat{q}}{(2\pi)^3} \phi_S(\hat{q})(\alpha_1 + \alpha_4 \hat{q}^2);$$

$$\beta_3 = \frac{8 e e_Q^2 N_S}{M^4 s} \int \frac{d^3\hat{q}}{(2\pi)^3} \phi_S(\hat{q}) |\hat{q}| \alpha_3$$

(effective coupling of bound states with $\gamma^*$)

$$\phi_S(1P, \hat{q}) = \sqrt{\frac{2}{3}} \frac{1}{\pi^{3/4}} \frac{1}{\beta_S^{5/2}} |\hat{q}| e^{-\frac{\hat{q}^2}{2\beta_S^2}},$$

$$\phi_S(2P, \hat{q}) = \sqrt{\frac{5}{3}} \frac{1}{\pi^{3/4}} \frac{1}{\beta_S^{5/2}} |\hat{q}| (1 - \frac{2\hat{q}^2}{5\beta_S^2}) e^{-\frac{\hat{q}^2}{2\beta_S^2}}$$

(3D Radial wave functions)

$$\Psi_s(\hat{q}) = N_S\left[\frac{M^2}{m^2} + i\frac{\gamma.\hat{q}}{m} + \cdots\right]\phi_s(\hat{q})$$

[PRD100, 054034 (2019); PRD97, 034021(2018)]

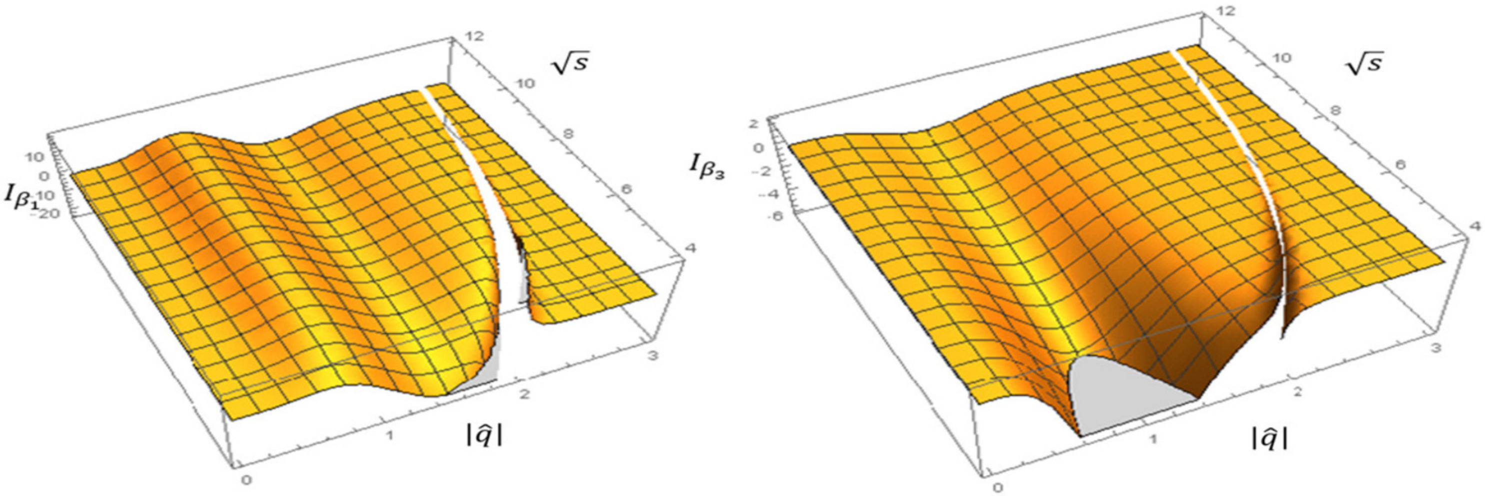
In meson rest frame:

$\frac{\gamma.\hat{q}}{m} \approx (\gamma.v)$ in NRQCD

$\Rightarrow$ $(\gamma.\hat{q})$/m reflects $O(v)$ correction to $M_{fi}$ (Consistent with NRQCD power counting!)

$\Rightarrow$ Covariant structure of BS wave function automatically generates expansion in powers of quark relative velocity, $v$ in NRQCD - Correction beyond LO contribution in NRQCD

# 2D Plots of Integrands of transition form factor: $\beta_1, \beta_3$ vs $|\hat{q}|$ and $\sqrt{s}$




Regions of discontinuities in hadron internal momentum, $|\hat{q}|$ (in GeV) in the integrands $I_{\beta_1}$, and $I_{\beta_3}$ of form factors $\beta_1$, and $\beta_3$ respectively, in Eq. (25) at different center of mass energies, $\sqrt{s}$ (in GeV) in the range, 4.6 – 12 GeV for the process $e^- e^+ \rightarrow \gamma \chi_{c0}(1P)$.

| $\sqrt{s}$ | $|\hat{q}|$ (for $\beta_1$) | $|\hat{q}|$ (for $\beta_3$) |
|---|---|---|
| 12 | $2.79 < |\hat{q}| < 2.82$ | $2.79 < |\hat{q}| < 2.82$ |
| 10.6 | $2.73 < |\hat{q}| < 2.76$ | $2.73 < |\hat{q}| < 2.76$ |
| 8 | $2.54 < |\hat{q}| < 2.57$ | $2.54 < |\hat{q}| < 2.57$ |
| 6 | $2.28 < |\hat{q}| < 2.32$ | $2.27 < |\hat{q}| < 2.33$ |
| 4.6 | $2.01 < |\hat{q}| < 2.05$ | $2.01 < |\hat{q}| < 2.05$ |

(Narrow discontinuous regions in both $|\hat{q}|$ and $\sqrt{s}$ in integrands of $\beta_1, \beta_3$ -results of $Md\sigma$ integrations over the poles of quark propagators).

Discontinuities become narrower with increase in $\sqrt{s}$

- $\beta_1$ and $\beta_3$ encapsulate entire momentum dependence + play crucial role in maintaining gauge invariance of transition amplitude.
- Discontinuities presently ignored to ensure numerical stability

## Cross section for $e^-e^+ \to \gamma^* \to \gamma + \chi_{c0,}$

[Nucl. Phys. A104, 122783 (2024)]

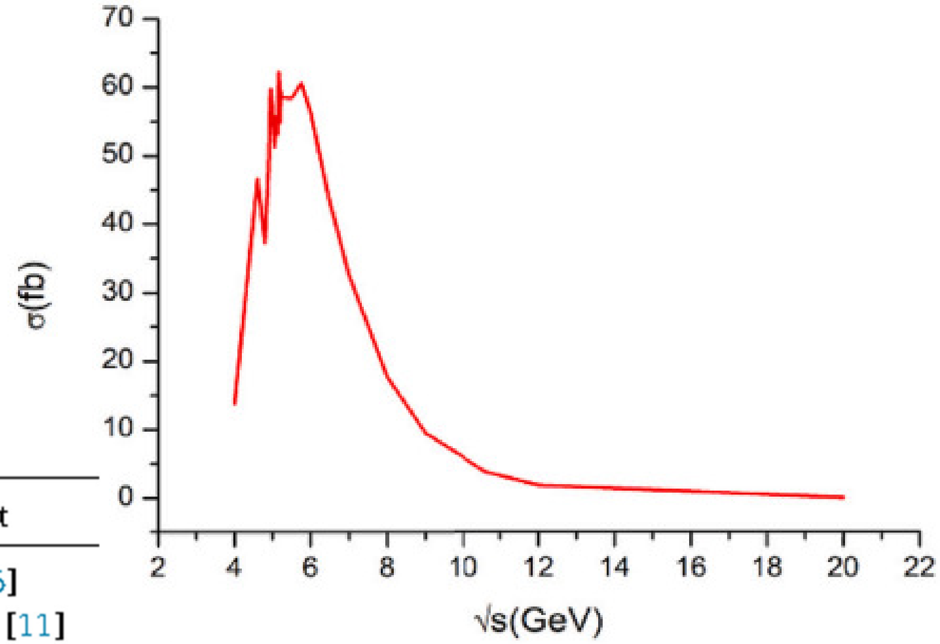
$$|\bar{M}_{fi}|^2 = \left[2\beta_1^2(-s+3m_e^2) - \frac{1}{2}\beta_3^2 s^2(1-cos^2\theta)\right]$$

$$\sigma = \frac{1}{32\pi^2 s^{3/2}}|\vec{P}'|\int d\Omega' |\bar{M}_{fi}|^2$$

Cross sections (in fb) at $\sqrt{s}$ =10.6GeV- 4 GeV in BSE

| Process | $\sqrt{s}$ | BSE | Experiment |
|---|---|---|---|
| $e^-e^+ \to \gamma\chi_{c0}(1P)$ | 10.6 | 3.810 | $<205.9$ [16] |
| $e^-e^+ \to \gamma\chi_{c0}(1P)$ | 4.6 | 46.617 | $<2.6\times10^3$ [11] |
| $e^-e^+ \to \gamma\chi_{c0}(1P)$ | 4.0 | 13.785 | $<4.5\times10^3$ [11] |
| $e^-e^+ \to \gamma\chi_{c0}(2P)$ | 10.6 | 3.570 | |
| $e^-e^+ \to \gamma\chi_{c0}(2P)$ | 4.6 | 64.073 | |

($\sigma$ within upper limits of Belle and BESIII)




$\sigma$(s) ~ $1/s^2$ at high energies

Experiment:
[16] S. Jia et al.,[Belle], PRD98 (2018)
[11] M. Ablikim, et al., [BESIII] PRD104, (2021)

Other models (at 10.6 GeV):

$\sigma(e^-e^+ \to \gamma^* \to \gamma + \chi_{c0,}$(1P))= $3.11^{+1.05}_{-0.94}$ fb [Chung, JHEP 09, 195 (2021)]

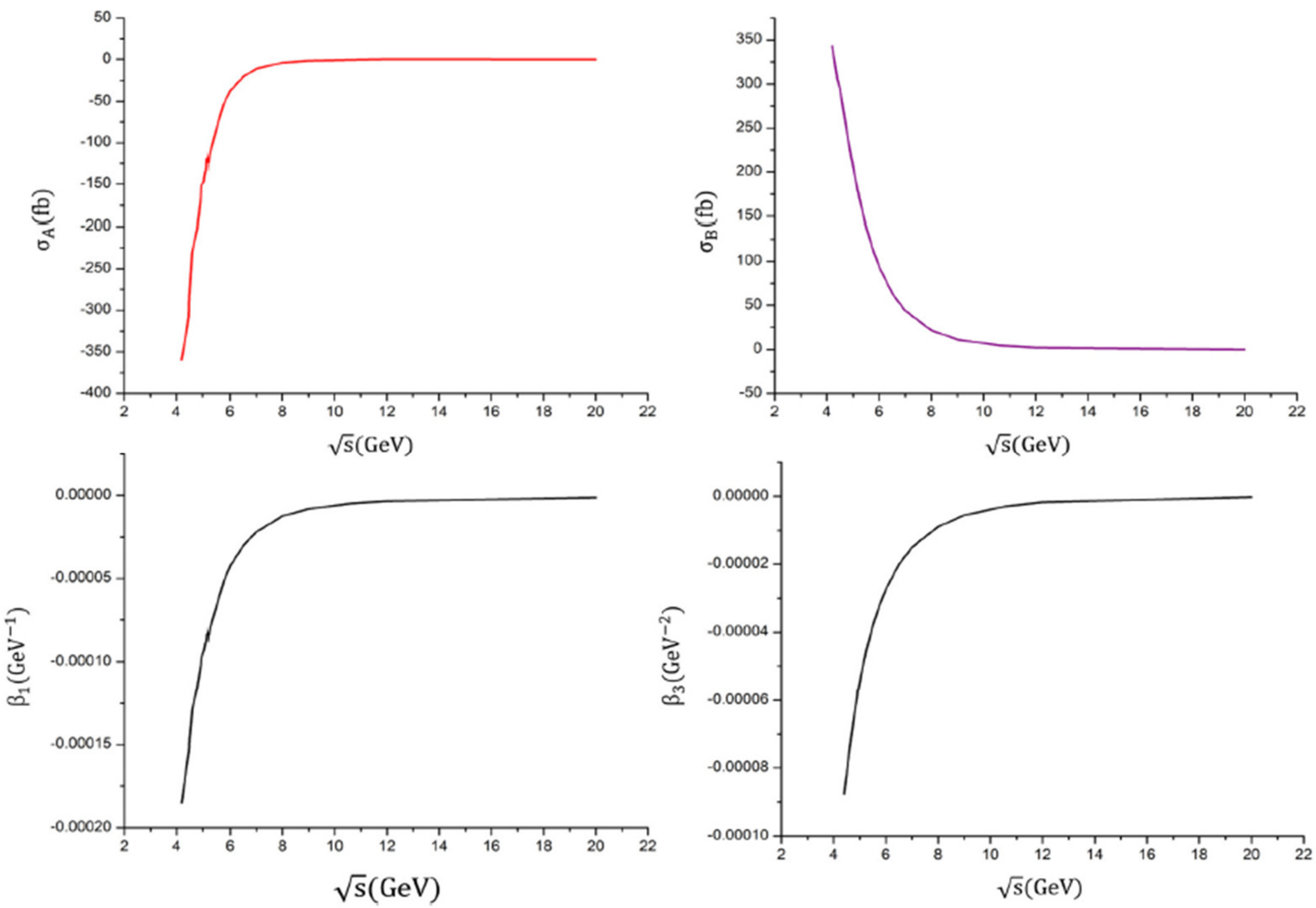
$\sigma(e^-e^+ \to \gamma^* \to \gamma + \chi_{c0,}$(1P))=1.885 fb [Chen et al., PRD 88, 074021 (2013)]

| Process | $\sqrt{s}$ | $\beta_1[GeV^{-1}]$ | $\beta_3[GeV^{-2}]$ |
|---|---|---|---|
| $e^-e^+ \rightarrow \gamma + \chi_{c0}$(1P) | 10.6 | -4.869* $10^{-6}$ | -2.93*$10^{-6}$ |
| $e^-e^+ \rightarrow \gamma + \chi_{c0}$(1P) | 4.6 | -1.242*$10^{-4}$ | -7.517*$10^{-5}$ |
| $e^-e^+ \rightarrow \gamma + \chi_{c0}$(2P) | 10.6 | 5.583*$10^{-6}$ | 2.967*$10^{-6}$ |
| $e^-e^+ \rightarrow \gamma + \chi_{c0}$(2P) | 4.6 | 1.473*$10^{-4}$ | 6.345*$10^{-5}$ |

(Form factors: $\beta_1$ , $\beta_3$ )

$$|\bar{M}_{fi}|^2 = \left[2\beta_1^2(-s+3m_e^2) - \frac{1}{2}\beta_3^2 s^2(1-\cos^2\theta)\right]$$

$$\sigma = \sigma_A + \sigma_B;$$



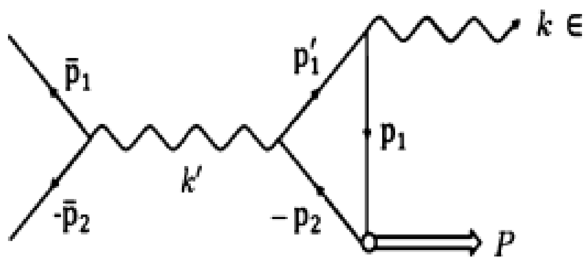

4-5 GeV: $\beta_1$ drives growth of cross section with oscillations due to rapid change.

5-7 GeV: "Interference" between $\sigma_A$ and $\sigma_B$ leading to sharp fluctuations.

7-20 GeV: $\beta_3$ dominates leading to smooth fall in cross section.

$e^- + e^+ \longrightarrow \gamma* \longrightarrow \gamma + \chi_{c1}$

Nucl. Phys. A1041, 122783 (2024)




+ Exchange

$M_{fi}$ expressed as: $M_{fi}=ie_Q^2[\overline{v_2}\gamma_\mu u_1]\, M_\mu$ ;

$M_\mu = <\gamma\chi_{c1}\,|J_\mu|0>$

$$M_{fi} = [\bar{v}^{s_2}(p_2)\gamma_\mu u^{s_1}(p_1)][g_2\epsilon_{\mu\nu\alpha\beta}\epsilon_\nu^\lambda\epsilon_\alpha^{\lambda'}k_\beta + g_3(I.\epsilon)\epsilon_{\mu\nu\alpha\beta}P_\nu\epsilon_\alpha^{\lambda'}k_\beta].$$ (Gauge-invariant form)

$$g_2 = \frac{8ee_Q^2N_A}{M^4s}\int\frac{d^3\hat{q}}{(2\pi)^3}\phi_A(\hat{q})\Theta_1,$$ (Form factors)

$(g_{2,3}\approx\frac{1}{s})$

$$g_3 = \frac{8ee_Q^2N_A}{M^4s}\int\frac{d^3\hat{q}}{(2\pi)^3}\phi_A(\hat{q})|\hat{q}|\Theta_6.$$

$$\Theta_1 = \theta_1(M-2\omega)I_1' + \rho_1(M+2\omega)I_1''$$

$$\Theta_6 = \theta_6(M-2\omega)I_1' + \rho_6(M+2\omega)I_1''.$$

(I′, I″ are results of $Md\sigma$ integrations over poles of quark propagators)

$$\Psi_A(\hat{q}) = N_A\gamma_5\left[iM\not{\epsilon} + \not{\epsilon}\not{P} + 2i\frac{\not{\epsilon}\not{P}\hat{\not{q}}}{M}\right]\phi_A(\hat{q})$$

(3D BS wave function under CIA)

$$\phi_A(1P,\hat{q}) = \sqrt{\frac{2}{3}}\frac{1}{\pi^{3/4}}\frac{1}{\beta_A^{5/2}}|\hat{q}|e^{-\frac{\hat{q}^2}{2\beta_A^2}},$$

$$\phi_A(2P,\hat{q}) = \sqrt{\frac{5}{3}}\frac{1}{\pi^{3/4}}\frac{1}{\beta_A^{5/2}}|\hat{q}|(1-\frac{2\hat{q}^2}{5\beta_A^2})e^{-\frac{\hat{q}^2}{2\beta_A^2}}$$

(3D radial wave functions)

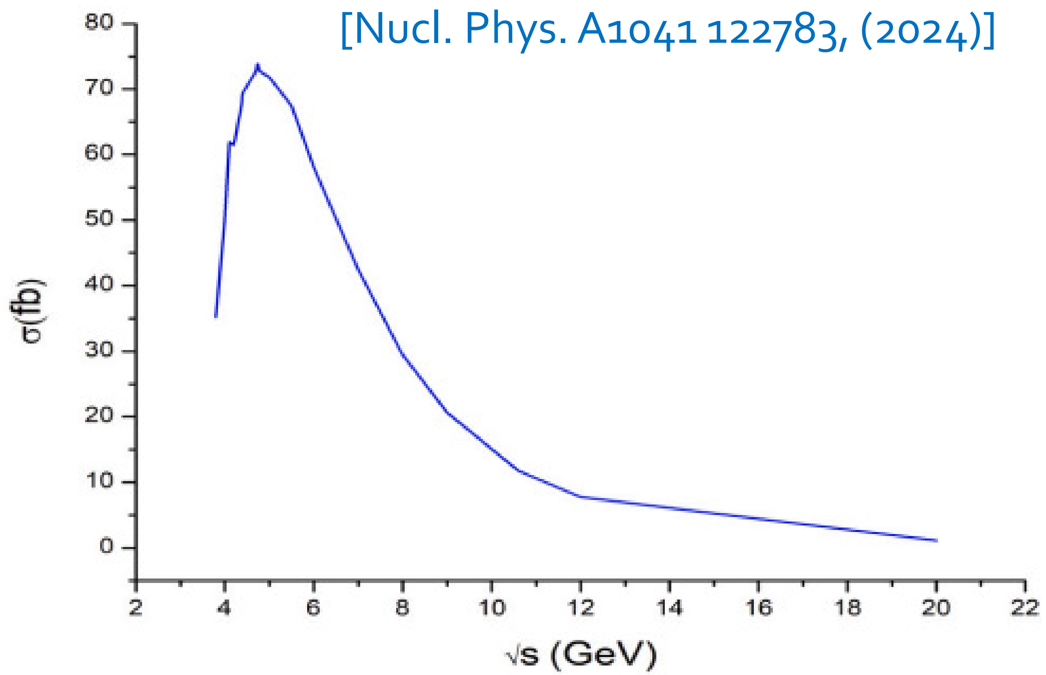
[PRD100, 054034 (2019); PRD97, 034021(2018); PRD102, 094024(2020)]

# Cross section (in fb) for $e^- e^+ \to \gamma^* \to \gamma + \chi_{c1}$

| Process | $\sqrt{s}$ | BSE | Experiment [11,16] |
|---|---|---|---|
| $e^-e^+ \to \gamma\chi_{c1}(1P)$ | 10.6 | 12.071 | $17.3^{+4.2}_{-3.9} \pm 1.7$ |
| $e^-e^+ \to \gamma\chi_{c1}(1P)$ | 4.6 | 75.832 | $(1.7^{+0.8}_{-0.6} \pm 0.2) \times 10^3$ |
| $e^-e^+ \to \gamma\chi_{c1}(1P)$ | 4.0 | 50.055 | $(4.5^{+1.5}_{-1.3} \pm 0.4) \times 10^3$ |
| $e^-e^+ \to \gamma\chi_{c1}(2P)$ | 10.6 | 9.751 | |
| $e^-e^+ \to \gamma\chi_{c1}(2P)$ | 4.6 | 43.518 | |

Experiment:
[11] M. Ablikim, et al.,( BESIII), PRD 104 (2021) 092001
[16] S. Jia, et al., (Belle) PRD98 (2018) 092015.

[Nucl. Phys. A1041 122783, (2024)]




Other models (at 10.6 GeV):
$\sigma = 9.7\ fb$ [Li et al., JHEP 01 (2014) 022]
$\sigma = 25.96\ fb$ [Sang, et al., JHEP 10 (2020) 098]

At 10.6 GeV: $\sigma$ =12.071fb (within 1.2σ of Belle measurements)

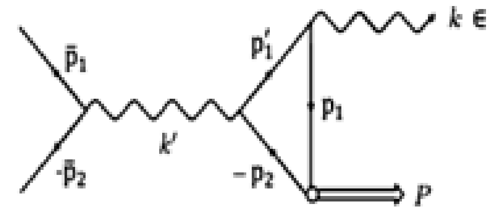
At 4.6GeV: σ=75.832fb (within 2σ of BESIII measurement).

At 4GeV: σ=50.055fb (within 2.8σ of BESIII measurement).

Minor fluctuations observed in plot of σ at BESIII energies, which are analyzed in terms of plots of form factors, g2 and g3

## $e^- e^+ \to \gamma^* \to \gamma + \eta_c$

[Nucl. Phys. A 1041 (2024) 122783]



+ Exchange

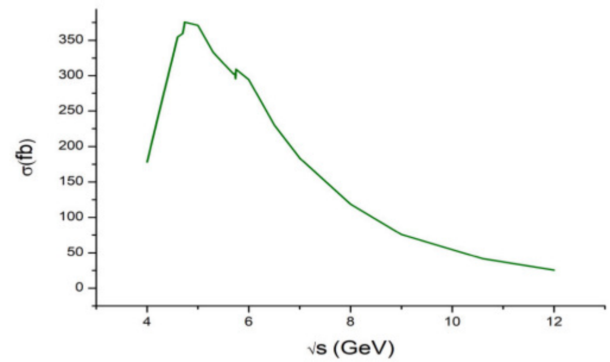
$$M_{fi} = [\bar{v}^{s_2}(p_2)\gamma_\mu u^{s_1}(p_1)]\beta\epsilon_{\mu\nu\alpha\beta}P_\nu\epsilon^{\lambda'}_\alpha k_\beta;$$ (Gauge-invariant form)

$$\beta = \frac{8ee_Q^2 N_P}{M^4 s}\int\frac{d^3\hat{q}}{(2\pi)^3}X_2'\phi_P(\hat{q});$$

$$X_2' = -b_1'(M-2\omega)I_1' + b_2'(M+2\omega)I_1''.$$

$$|\bar{M}_{fi}|^2 = \frac{1}{4}\beta^2[16m_e^2M^4 + 4m_e^2s^2 + s^3(1+cos^2\theta) - M^2(16m_e^2 s + s^2(1+3cos^2\theta))]$$

$$\sigma = \frac{1}{32\pi^2 s^{3/2}}|\vec{P}'|\int d\Omega'|\bar{M}_{fi}|^2$$




| Process | $\sqrt{s}$ | BSE | Experiment [13,16] | [53] |
|---|---|---|---|---|
| $e^-e^+ \to \gamma\eta_c(1S)$ | 10.6 | 41.4575 | $< 21.1$ | 70 |
| $e^-e^+ \to \gamma\eta_c(1S)$ | 4.6 | 310.449 | $(0.23 \pm 0.53 \pm 0.35) \times 10^3$ | 678 |
| $e^-e^+ \to \gamma\eta_c(1S)$ | 4.0 | 178.271 | $(0.44 \pm 1.02 \pm 0.32) \times 10^3$ | 540 |
| $e^-e^+ \to \gamma\eta_c(2S)$ | 10.6 | 22.371 | | 32 |
| $e^-e^+ \to \gamma\eta_c(2S)$ | 4.6 | 19.702 | | 33 |

Experiment:

[13] M. Ablikim, et al., PRD 96 (2017) 051101(R).

[16] S. Jia, et al., Belle Collaboration, PRD 98 (2018) 092015.

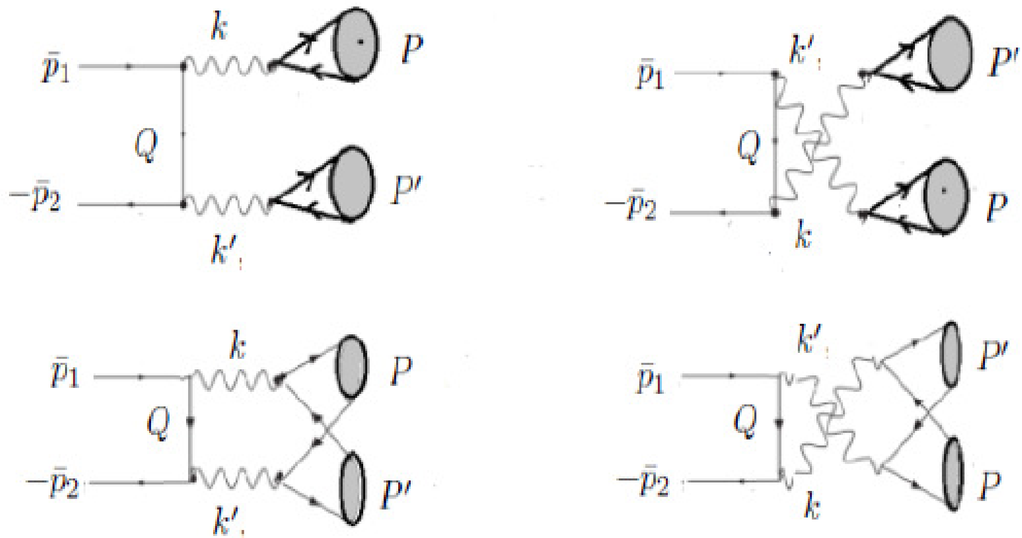
Other Models:

[53] Y. J. Li et al., JHEP 01,022 (2014)

$\sigma = 41.6 \pm 14.1\ fb$ (at 10.6 GeV) [Braguta, PRD 82, 074009 (2010)]

$e^- + e^+ \to \gamma^* + \gamma^* \to J/\psi + J/\psi$

[Nucl. Phys. A1053 (2025) 122969]




(Photon fragmentation)
$k^2 \sim M^2_{J/\psi}$

(Quark rearrangement)
$k^2 \sim s/4$

$J/\psi + J/\psi$ cross section suppressed by $\alpha^2/\alpha_s^2$ compared to $J/\psi + \eta_c$.

Fragmentation diagrams enhanced by $\frac{s^2}{16M^4}$ compared to rearrangement diagrams

$\to \sigma(J/\psi + J/\psi) > \sigma(J/\psi + \eta_c)$

Calculated cross section in BSE with diagrams $\approx 0(\alpha^4)$

At $\sqrt{s} = 10.6$ GeV, lowest order QCD calculation in $\alpha_s$ in BSE is found sufficient

- G. T. Bodwin, J. Lee, and E. Braaten, PRL 90. 162001(2003); PRD 67, 054023(2003).

- Let P, q, ε (P′, q′, ε′) – external (internal) momenta, and polarization vectors of two out- going vector mesons.
- Quark momenta:

$$p_1 = \frac{1}{2}P + q, \quad p_2 = \frac{1}{2}P - q, \quad p_1' = \frac{1}{2}P' - q', \qquad p_2' = \frac{1}{2}P' + q'.$$

- Fragmentation: Internal momenta q and q′ are independent:

$$k^2 k'^2 = M^2\, M'^2$$

- Quark-rearrangement: q and q′ are not independent
- => Photon propagators: $1/k^2$, $1/k'^2$;

  [$k = \frac{1}{2}$(P+P′)+q-q′; $k = \frac{1}{2}$(P+P′)-q+q′]

  depend upon q and q′.

Heavy-quark approximation: $m_Q$>> q ~ ($\alpha_s m_Q$ ), photon propagators are made independent of internal hadron momentum:

$$k^2 k'^2 \approx \frac{s^2}{16}$$

Amplitudes:

$$M_{fi}^{frg.} = \frac{e^2 e_Q^2}{M^2 M'^2}[\bar{v}_2 \gamma_\nu u_1]\frac{-i\gamma.Q+m_e}{Q^2+m_e^2}\int \frac{d^4q}{(2\pi)^4}\mathrm{Tr}[\gamma_\mu \overline{\Psi}(P,q)] \int \frac{d^4q'}{(2\pi)^4}\mathrm{Tr}[\gamma_\mu \overline{\Psi}(P',q')]$$

+(Exchange);

$$M_{fi}^{rearr.} = \frac{16 e^2 e_Q^2}{s^2}[\bar{v}_2 \gamma_\nu u_1]\frac{-i\gamma.Q+m_e}{Q^2+m_e^2}\int \frac{d^4q}{(2\pi)^4}\int \frac{d^4q'}{(2\pi)^4}\mathrm{Tr}[\gamma_\mu \overline{\Psi}(P',q')\gamma_\nu \overline{\Psi}(P,q)]$$

+(Exchange)

Reduction to 3D form:

$d^4$q $=d^3\hat{q}$ M d$\sigma$; $\overline{\Psi}(\hat{q}) = \int M d\sigma\, \overline{\Psi}(P,q)$;

$$\bar{\Psi}_V(\hat{q}) = \left[iM\not{\epsilon} + (\hat{q}.\epsilon)\frac{M}{m} + \not{P}\not{\epsilon} + \frac{i}{2m}\not{\hat{q}}\not{\epsilon}\not{P} + \frac{i}{2m}(\hat{q}.\epsilon)\not{P}\right]\phi_V(\hat{q}),$$

$$\bar{\Psi}_V(\hat{q}') = \left[iM'\not{\epsilon}' + (\hat{q}'.\epsilon')\frac{M'}{m} + \not{P}'\not{\epsilon}' + \frac{i}{2m}\not{\hat{q}}'\not{\epsilon}'\not{P}' + \frac{i}{2m}(\hat{q}'.\epsilon')\not{P}'\right]\phi_V(\hat{q}').$$

(3D BS wave functions)

$$\phi_V(1S,\hat{q}) = e^{-\frac{\hat{q}^2}{2\beta^2}};$$

$$\phi_V(2S,\hat{q}) = (1 - \frac{2\hat{q}^2}{3\beta^2})e^{-\frac{\hat{q}^2}{2\beta^2}}.$$

(Unnormalized 3D Radial wave functions* of final vector mesons obtained as solutions of their mass spectral equations)

*[ PRD100, 054034 (2019)]

$M_{fi}$ expressible in terms of 3D integrals involving two final vector mesons:

$$G_1 = \int \frac{d^3\hat{q}}{(2\pi)^3} \phi_V(\hat{q})$$

$$G_1' = \int \frac{d^3\hat{q}}{(2\pi)^3} |\hat{q}| \phi_V(\hat{q})$$

$$G_2 = \int \frac{d^3\hat{q}'}{(2\pi)^3} \phi_V(\hat{q}')$$

$$G_2' = \int \frac{d^3\hat{q}'}{(2\pi)^3} |\hat{q}'| \phi_V(\hat{q}')$$

Leptonic decay constants:

$$f_{V1} = 4\sqrt{3} N_{V1} \int \frac{d^3\hat{q}}{(2\pi)^3} \phi_{V1}(\hat{q}) = 4\sqrt{3} N_{V1} G_1 \;;$$

$$f_{V2} = 4\sqrt{3} N_{V2} \int \frac{d^3\hat{q}'}{(2\pi)^3} \phi_{V2}(\hat{q}') = 4\sqrt{3} N_{V2} G_2$$

Form factors:

| Meson | $N_V\,[GeV^{-2}]$ | $G\,[GeV^3]$ | $G'\,[GeV^4]$ | $f_V\,[GeV]$ |
|---|---|---|---|---|
| $J/\psi$ | 7.2958 | 0.00826 | 0.00668 | 0.4175 (Exp.=0.411) |
| $\psi(2S)$ | 5.9281 | 0.00882 | 0.00729 | 0.3622 (Exp.=0.279) |

$$|\overline{M_{fi}}|^2 = \frac{1}{4} \Sigma_{s1,s2,\lambda,\lambda'} M_{fi}^{\dagger} M_{fi} = |\overline{M_{fi}}|^2{}_{Frag.} + |\overline{M_{fi}}|^2{}_{Rearr.} + |\overline{M_{fi}}|^2{}_{Interf.}$$

$$\sigma = \frac{1}{32\pi^2 s^{3/2}} |\vec{P}'| \int d\Omega' |\bar{M}_{fi}|^2 \qquad |\vec{P}'| = \sqrt{\frac{1}{s}[s-(M+M')^2][s-(M-M')^2]}$$

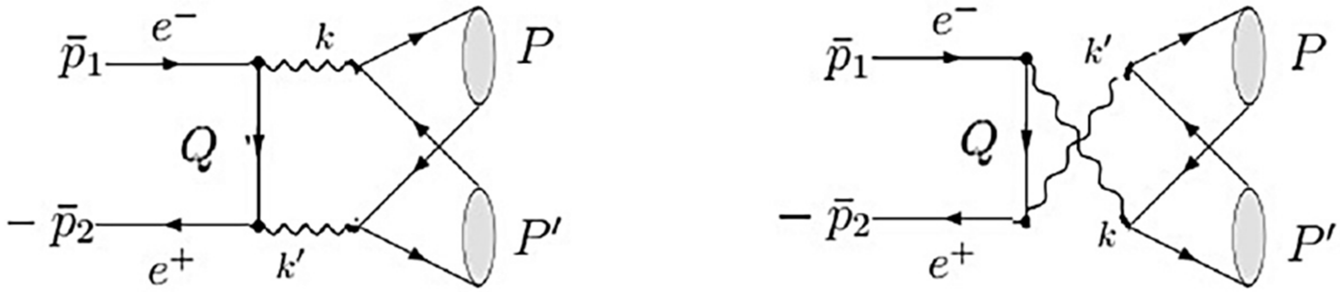
**Table 1**
Cross sections (in fb) at leading order (LO) for process, $e^- e^+ \to J/\Psi J/\Psi$ calculated in present work at $\sqrt{s} = 10.6 GeV$ along with experimental data, and results of other models.

| Process | BSE-CIA | Expt. [2] | NRQCD [5] | [14] | [4] |
|---|---|---|---|---|---|
| $e^- e^+ \to J/\psi J/\psi$ | 2.4549 | $< 9.1$ | 6.65±3.02 | 2.260 | 2.12±0.85 |
| $e^- e^+ \to J/\psi \psi(2S)$ | 1.969 | $< 13.3$ | 5.52±2.50 | 1.460 | 1.43±0.57 |
| $e^- e^+ \to \psi(2S)\psi(2S)$ | 1.6281 | $< 5.2$ | 1.15±0.52 | 0.230 | 0.24±0.10 |

| $\boldsymbol{\sigma}(e^- e^+ \to J/\psi\ J/\psi)$* | $\sigma^{Frag.}$(% of $\sigma$) | $\sigma^{Arrn.}$(% of $\sigma$) | $\sigma^{Int}$(% of $\sigma$) |
|---|---|---|---|
| 2.4549 fb | 3.215 fb (131%) | 0.3639 fb (14.823%) | -1.124 fb (-45.814%) |


*Nucl. Phys. A1053, 122969 (2025).
[2] (Belle), PRD70, 071102 (2004).
[5] PRL 90, 162001 (2003).
[4] PRD 78, 054025 (2008).
[14] Phys. At. Nucl. 67, 1338 (2004).

## $e^- e^+ \to \gamma^* \gamma^* \to \eta_c + \eta_c$




$$M_{fi}^{rearr.} = \frac{16e^2 e_Q^2}{s^2}[\bar{v}_2 \gamma_\nu u_1]\frac{-i\gamma.Q+m_e}{Q^2+m_e^2}\int \frac{d^4 q}{(2\pi)^4}\int \frac{d^4 q'}{(2\pi)^4}\mathrm{Tr}[\gamma_\mu \overline{\Psi}(P',q')\gamma_\nu \overline{\Psi}(P,q)] + (\mathrm{Exchange})$$

Reduction to 3D form:

$d^4 q = d^3\hat{q}\, M d\sigma$; $\overline{\Psi}(\hat{q}) = \int M d\sigma\, \overline{\Psi}(P,q)$;

$$\bar{\Psi}_P(\hat{q}) = N_P \gamma_5 [M - i\not{P} + \frac{2\not{q}\not{P}}{M}]\phi_P(\hat{q})$$

$$M_{fi} = \frac{16e^2 e_Q^2}{s^2} N_P N_P' \Big[ 2G_1 G_2(-4MM' + 4P.P')[\bar{v}^{(s2)}(\bar{p}_2)[\frac{i\not{Q} + 4m_e}{Q^2 + m_e^2}]u^{(s1)}(\bar{p}_1)] +$$

$$4(-4G_1 G_2 - \frac{16I.I'}{MM'}G_1' G_2')[\bar{v}^{(s2)}(\bar{p}_2)\not{P}\frac{-i\not{Q} + m_e}{Q^2 + m_e^2}\not{P}' u^{(s1)}(\bar{p}_1)] - \frac{16P.P'}{MM'}G_1' G_2'[\bar{v}^{(s2)}(\bar{p}_2)\not{q}\frac{-i\not{Q} + m_e}{Q^2 + m_e^2}\not{q}' u^{(s1)}(\bar{p}_1)\Big]$$

$M_{fi}^{rearr.}$ again expressed in terms of 3D integrals, G, G′, with G related to $f_P$

Cross sections (in fb) at leading order (LO) for process, $e^- e^+ \to \eta_c \eta_c$ calculated in present work at $\sqrt{s} = 10.6 GeV$ along with results of other models.

| Process | BSE-CIA | NRQCD [5] |
|---|---|---|
| $e^- e^+ \to \eta_c(1S)\eta_c(1S)$ | $2.087 \times 10^{-3}$ | $(1.83 \pm 0.10) \times 10^{-3}$ |
| $e^- e^+ \to \eta_c(2S)\eta_c(1S)$ | $1.8087 \times 10^{-3}$ | $(1.52 \pm 0.08) \times 10^{-3}$ |
| $e^- e^+ \to \eta_c(2S)\eta_c(2S)$ | $0.2369 \times 10^{-3}$ | $(0.31 \pm 0.02) \times 10^{-3}$ |

Form factors:

| Meson | $G\ [GeV^3]$ | $G'\ [GeV^4]$ | $f_P\ [GeV]$ |
|---|---|---|---|
| $\eta_c(1S)$ | 0.00812 | 0.00652 | 0.3282 |
| $\eta_c(2S)$ | -0.00877 | -0.01206 | 0.2363 |

## Leptonic decay constants:

$$f_P = 4\sqrt{3} N_P \int \frac{d^3\hat{q}}{(2\pi)^3} \phi_P(\hat{q}) = 4\sqrt{3} N_P\, G$$


* Nucl. Phys. A1053, 122969 (2025).

[5] G.T. Bodwin, J. Lee, E. Braaten, PRL 90, 162001 (2003).

# Conclusion

- Studied the production processes at B-factories at 10.6 GeV for:

  (A) $e- + e+ \longrightarrow \gamma* \longrightarrow \gamma$ + H ; (H= $\chi_{c0}$, $\chi_{c1}$, and $\eta_c$)

  (B) $e- + e+ \longrightarrow \gamma* + \gamma* \longrightarrow$ J/ψ + J/ψ; and $e^-e^+ \to \gamma^*\gamma^* \to \eta_c + \eta_c$

  in framework of BSE under Covariant Instantaneous Ansatz.

- In (A), Form factors $\beta_1$ , $\beta_3$ etc. absorb the entire momentum dependence in the amplitudes and cross section.
- In (B), $M_{fi}$ expressed in terms of 3D integrals, G - related to leptonic decay constant of the meson.
- Substantial contributions to cross sections from lowest order diagrams in BSE. Comparison made with data and other models.
- Cross sections analysed in terms of form factors.
- Percentage contribution from fragmentation and rearrangement diagrams, along with interference of these amplitudes calculated for (B).

Based on:

S.Bhatnagar, Talk at 21st Intl. conf. on Hadron Spectroscopy and Structure (Hadron 2025), Osaka University (March 27-31, 2025).

S.Bhatnagar, H.Negash, Nucl. Phys. A1041, 122783 (2025).

S.Bhatnagar, V.Guleria, Nucl. Phys. A1053, 122969 (2024).

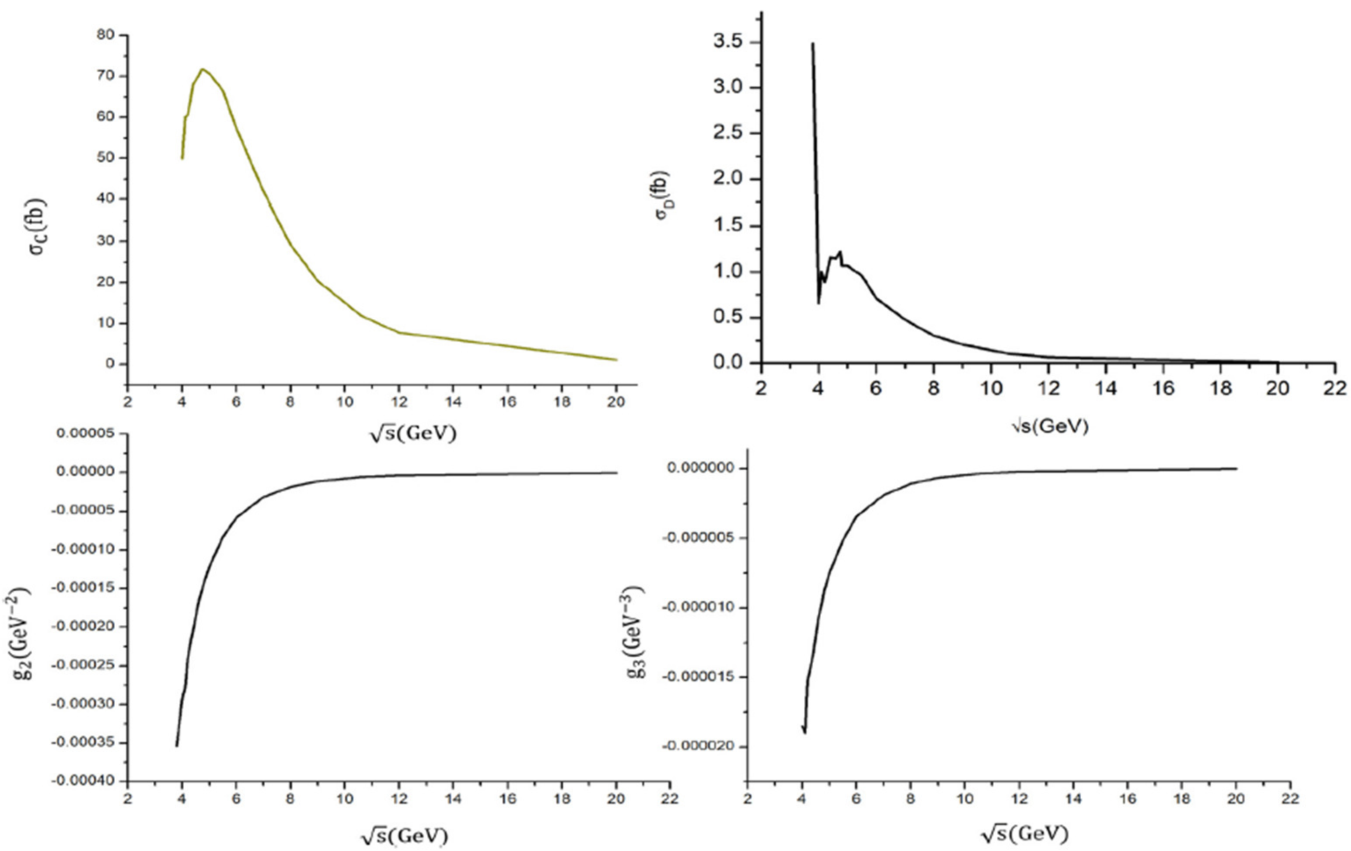
# Thank you

*Cross section*:

$$\sigma = \frac{1}{32\pi^2 s^{3/2}} |\vec{P}'| \int d\Omega' |\bar{M}_{fi}|^2, \qquad \sigma = \sigma_C + \sigma_D$$

$$|\bar{M}_{fi}|^2 = \frac{1}{4}\left[\frac{1}{48M^2} g_2^2 [4(M^4 + s^2)(-6m_e^2 + s) + s^3(1+cos\theta)^2 + M^2(48m_e^2 s - 8s^2 - s^2(1+cos\theta)(-1+3cos\theta))] - \frac{1}{96}\frac{g_3^2}{3}(M^2 - s)[64m_e^2 s + s^2(5+3cos^2\theta) + M^2(-64m_e^2 + 6s(1+cos\theta))]\right].$$

(Form factor plots)




(Rapid fluctuations in $\sigma_D$ in 4.0< $\sqrt{s}$ < 6 GeV due to fluctuations in $g_3$).

But $\sigma_D$ has negligible contribution (1.30 - 1.46%) to total $\sigma$ in any energy interval, but introduces mild fluctuations at low energy.