Boson Stars

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Talk is based on our Publications: CQG(2014), GERG(2015), PRD(Rapid)(2016), PRD(2016), PLB(2017), FYBS(2018), EPJC(2019), PoS(2020), CQG(2025).

Boson Stars: some Basics and Motivations:

- **Field theoretical models of complex scalar fields coupled with gravity provide the nonsingular asymptotically flat solutions referred to as the boson stars: the gravitational bound states of scalar particles.**
- **In GR, "a nonsingular asymptotically flat solution" refers to a solution that describes a spacetime which is free from singularities i.e., no points of infinite curvature at any point, while still exhibiting the property of being asymptotically flat**
- (it means that the geometry approaches the flat spacetime of SR at very large distances from the source of gravity).
- **Boson stars are** *hypothetical* **astronomical objects that consist of particles called bosons.**
- Scalar fields occur routinely in particle physics, field theory, gravity theory, string theory and cosmology.

Boson Stars: some Basics and Motivations:

- **Scalar fields interacting with gravity, through a gravitational condensation mechanism, might form spherically symmetric objects: the boson stars that would be stopped from collapsing by HUP** (cf. Jochum van der Bij, M. Gleiser PLB (1987)).
- Boson stars were concieved almost half a century ago and they are very interesting from a theoretical point of view.
- Their potential astrophysical applications are of wide interest :
- Accretion disks around boson stars have been studied.
- Gravitational-wave signatures of boson stars have been studied.
- Binary systems of boson stars have also been studied.
- Discovery of hairy black holes based on boson stars have been studied. It has also been discovered that rotating boson stars allow for static orbits.

Boson Stars: some Basics and Motivations:

- **Oldenburg group has been at the fore front of many of these studies!**
- Just like super massive black holes, boson stars could even exist at the centre of galaxies and could possibly explain many of the observed properties of active galactic core.
- Boson stars could have been formed e.g., through gravitational collapse during the primordial stages of the big bang.
- It may be possible to detect them by the gravitational radiation emitted by a pair of co-orbiting boson stars.
- Gravitational lensing by boson stars has also been considered.
- Boson stars have also been proposed as dark matter candidates.
- Boson stars have also found applications based on AdS/CFT conjecture.

Boson Stars: some Basics and Motivations:

- **Amongst the important properties of boson stars, their mass, their size and their particle number (which is proportional to their charge) are of great interest!**
- **Particle number of boson stars arises from the $U(1)$ symmetry of the underlying theory, which gives rise to the conserved current and the associated charge also when $U(1)$ symmetry is made a local symmetry, charged boson stars arise.**
- Boson star solutions, could in principle, be obtained by employing any self-interaction scalar field potential.
- These theories are quite interesting as they are perfectly well behaved from physical viewpoint and they lead to compactness of solutions.
- **Here compact is meant to describe a solution whose scalar field vanishes identically outside a compact region of space.**

Boson Stars: some Basics and Motivations:

- **Oldenburg group has made extensive studies of boson star solutions and concluded that the compact boson stars arise when a V-shaped self-interaction potential is employed.**
- NB: Boson stars in a theory *without* a conical potential do not feature sharp boundaries.
- **In compact boson stars, scalar field is completely confined to a finite region of space.**
- **Boson star solutions do not exist for arbitrary values of parameters. They exist only for a certain domain of parameters!**
- In our work, we first obtain the domain of existance of parameters of the theory for which boson star solutions exist and then study the various properties of boson star solutions.

Model-A: (Massive CSF, EMF, MC-Gravity, Λ)

- **Model-A: (CQG(2014), PRD(Rapid)(2016), PRD(2016)). (Collaboration: Sanjeev K, Usha K, Daya Shankar K)**

$$
\begin{aligned}
S &= \int \left[\frac{R - 2\Lambda}{16\pi G} + \mathcal{L}_M \right] \sqrt{-g}\; d^4 x \,, \\
\mathcal{L}_M &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - (D_\mu \Phi)^* (D^\mu \Phi) - V(|\Phi|) \,, \\
D_\mu \Phi &= (\partial_\mu \Phi + ieA_\mu \Phi) \,, F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) \,, \\
V(|\Phi|) &:= (m^2 |\Phi|^2 + \lambda |\Phi|)
\end{aligned}
$$

- Asterisk denotes complex conjugation, G is Newton's constant R is the Ricci curvature scalar, $g = det(g_{\mu\nu})$, $g_{\mu\nu}$ is metric tensor. Λ is cosmological constant (CC).
  1. $+$ve Λ implies boson stars in de Sitter (dS) space
  2. -ve Λ implies boson stars in Anti- de Sitter (AdS) space

- **Variational principle gives EoM . For the Model-A, EoM are:**

$$
\begin{aligned}
G_{\mu\nu} &\equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \, , \\
\partial_\mu \left(\sqrt{-g} F^{\mu\nu} \right) &= -i\, e \sqrt{-g} \left[\Phi^* (D^\nu \Phi) - \Phi (D^\nu \Phi)^* \right] , \\
D_\mu \left(\sqrt{-g} D^\mu \Phi \right) &= 2m^2 \sqrt{-g}\, \Phi + \frac{\lambda}{2} \sqrt{-g}\, \frac{\Phi}{|\Phi|} \, , \\
\left[D_\mu \left(\sqrt{-g} D^\mu \Phi \right) \right]^* &= 2m^2 \sqrt{-g}\, \Phi^* + \frac{\lambda}{2} \sqrt{-g}\, \frac{\Phi^*}{|\Phi|}
\end{aligned}
$$

- **The energy-momentum tensor $T_{\mu\nu}$ is given by**

$$
\begin{aligned}
T_{\mu\nu} = \Big[& \Big(F_{\mu\alpha} F_{\nu\beta}\; g^{\alpha\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \Big) \\
& + (D_\mu \Phi)^* (D_\nu \Phi) + (D_\mu \Phi)(D_\nu \Phi)^* \\
& - g_{\mu\nu} \left((D_\alpha \Phi)^* (D_\beta \Phi) \right) g^{\alpha\beta} - g_{\mu\nu}\; V(|\Phi|) \Big] \, .
\end{aligned}
$$

- **We assume that $A \equiv A(r)$ and $N \equiv N(r)$ (depend only on r).**
- **For spherically symmetric solutions we adopt a static spherically symmetric metric with Schwarzschild-like coordinates as:**

$$
\begin{aligned}
ds^2 &:= g_{\mu\nu} dx^\mu dx^\nu \\
&= \Big[-A(r)^2 N(r) dt^2 + N(r)^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2 \Big] \\
g_{\mu\nu} &:= diag\big(-A(r)^2 N(r) + N(r)^{-1} + r^2, r^2 sin^2\theta \big)
\end{aligned}
$$

- **Christoffel Symbol, Ricci Tensor, Ricci Scalar :**

$$
\begin{aligned}
\Gamma^\alpha_{\mu\nu} &:= \frac{1}{2} g^{\alpha\beta} \Big(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu} \Big) \\
R_{\mu\nu} &:= \Big(\partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\beta_{\mu\nu} \Gamma^\alpha_{\beta\alpha} - \Gamma^\beta_{\mu\alpha} \Gamma^\alpha_{\nu\alpha} \Big)
\end{aligned}
$$

$$
\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} = g_{\mu\nu} R^{\mu\nu} \\
&= -2N \left(\frac{A''}{A} + \frac{N''}{2N} + \frac{2N'}{rN} + \frac{3N'A'}{2NA} + \frac{2A'}{rA} + \frac{1}{r^2} - \frac{1}{r^2 N} \right)
\end{aligned}
$$

- Einstein Tensor: $G_{\mu\nu} := \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)$:
- Einstein Eq.: $G_{\mu\nu} := \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = (8\pi G T_{\mu\nu} - \Lambda \, g_{\mu\nu})$
- $\Longrightarrow$

$$G_t^t = (8\pi G \;\; T_t^t - \Lambda) \;\; , \quad (G_r^r = 8\pi G \;\; T_r^r - \Lambda) \;\; ,$$
$$G_\theta^\theta = (8\pi G \;\; T_\theta^\theta - \Lambda) \;\; , \quad G_\varphi^\varphi = (8\pi G \;\; T_\varphi^\varphi - \Lambda)$$

- The last 2 Eqs. yield identical results.
- **We make the Ansätze for matter fields as:**

$$\Phi(x^\mu) = \phi(r) e^{i\omega t} \quad , \quad A_\mu(x^\mu) dx^\mu = A_t(r) dt.$$

- **We work under the assumption of Vanishing EM field which implies: $\vec{B} = \nabla . \vec{A} = \mathbf{0}$.**
- This leaves us only with the time-like component of A_μ. Space-like components of A_μ do not contribute in this assumption.

Re-scaling of Field Variables and Const. Parameters

- **We introduce new Dim-less Const parameters as:**

$$\alpha = \frac{4\pi G\, m^2}{e^2}, \qquad \tilde{\lambda} = \frac{\lambda\, e}{\sqrt{2}\; m^3}, \qquad \tilde{\Lambda} = \frac{\Lambda}{m^2}.$$

- We introduce two fields $h(r)$ and $b(r)$ as:

$$h(r) = \frac{\sqrt{2}\; e\,\phi(r)}{m}, \qquad b(r) = \frac{\omega + eA_t(r)}{m}.$$

- **We then introduce a Dim-less coordinate $\hat{r}$ defined by:**

$$\hat{r} = m\, r \quad \Longrightarrow \quad \frac{d}{dr} = m\frac{d}{d\hat{r}}$$

- **This gives us Field variables with Dim-less arguments ($\hat{r}$ as:**

$$h(\hat{r}) := \frac{\sqrt{2}\; e\,\phi(\hat{r})}{m}, \qquad b(\hat{r}) = \frac{\omega + eA_t(\hat{r})}{m}.$$

- **After simplifying Einstein Eqs. and Matter Field Eqs., we obtain 4 Field Eqs. in the 4 fields: (h, b, A, N) to be solved numerically:**

$$
\begin{aligned}
h'' &= \Bigg[\frac{\alpha\,\hat{r}h'}{A^2N}\left(A^2h^2+2A^2h\tilde{\lambda}+\ b'^2\right)-\frac{h'\left(1+N-\tilde{\Lambda}\hat{r}^2\right)}{\hat{r}N} \\
&\quad +\frac{A^2Nh+A^2N\tilde{\lambda}\,\mathrm{sign}(h)-b^2h}{A^2N^2}\Bigg] \\
b'' &= \left[\frac{\alpha}{A^2N^2}\hat{r}b'\left(A^2N^2h'^2+b^2h^2\right)-\frac{2b'}{\hat{r}}+\frac{bh^2}{N}\right] \\
N' &= \Bigg[\frac{1-N-\tilde{\Lambda}\hat{r}^2}{\hat{r}}-\frac{\alpha\hat{r}}{A^2N}\Bigg(A^2N^2h'^2+Nb'^2+b^2h^2 \\
&\quad +A^2Nh^2+2A^2Nh\tilde{\lambda}\Bigg)\Bigg] \\
A' &= \left[\frac{\alpha\hat{r}}{AN^2}\left(A^2N^2h'^2+b^2h^2\right)\right]
\end{aligned}
$$

$$
\mathrm{sign}(h)=\begin{cases}\pm 1 & h>0,\ \ h<0\\ 0 & h=0\end{cases}
$$

- For obtaining BS solutions we choose the following BCs:

$$A(r_\mathrm{o}) = 1 \quad , \quad N(0) = 1 \quad , \quad b'(0) = 0 \quad ,$$
$$h'(0) = 0 \quad , \quad h(r_\mathrm{o}) = 0 \quad , \quad h'(r_\mathrm{o}) = 0$$

- For the boson stars we match the exterior region $\hat{r} > \hat{r}_o$, with the Reissner-Nordström (dS or AdS) solutions.

- For (globally regular) boson shell solutions with empty space-time in the interior of the shells, $r < r_i$, we choose the BC's as follows: For A(r), we choose the BC: $A(r_\mathrm{o}) = 1$ and also:

$$\begin{aligned} N(\hat{r}_i) &= 1 - \frac{\Lambda}{3}\hat{r}_i^2 , \quad b'(\hat{r}_i) = 0 \, , \quad h(\hat{r}_i) = 0 \, , \\ h'(\hat{r}_i) &= 0 \, , \qquad\qquad h(\hat{r}_o) = 0 \, , \quad h'(\hat{r}_o) = 0. \end{aligned}$$

Where $\hat{r}_i$ and $\hat{r}_o$ are the inner and outer radii of the shell. Here, we match the interior region $\hat{r} < \hat{r}_i$ and the exterior region $\hat{r} > \hat{r}_o$, with the appropriate Reissner-Nordström (dS or AdS) solutions.

Mass and Conserved Charge:

- U(1) invariance of theory leads to conserved Noether current density:

$$j^{\mu} = -i\,e\,\{\Phi(D^{\mu}\Phi)^{*} - \Phi^{*}(D^{\mu}\Phi)\} \quad , \quad j^{\mu}_{;\mu} = 0$$

- **Its time component corresponds to the charge density. The global charge Q of the boson star is:**

$$Q = -\frac{1}{4\pi}\int_{r_i}^{r_0} j^{t}\sqrt{-g}\,dr\,d\theta\,d\phi \quad , \quad j^{t} = -\frac{e\,h^{2}(r)b(r)}{A^{2}(r)N(r)}$$

- Mass M of the boson star solutions is:

$$M = \left(1 - N(r_o) + \frac{\alpha Q^{2}}{r_o^{2}} - \frac{\Lambda}{3}r_o^{2}\right)\frac{r_o}{2}$$

- Theories in dS space with $+$ ve Λ are relevant from observational point of view as the observations seem to indicate existence of a positive CC. (- useful in models of dark energy of the universe).
- Theories in AdS space with - ve Λ are meaningful in AdS/CFT theories (cf. Maldacena (1997), Witten (1998), Brodsky (2011).

Phase diagrams and Bifurcation phenomenon!

- **In our studies of the phase diagrams of boson stars, we find that a passage from one set of phase trajectories to another occurs suddenly or catastrophically.**

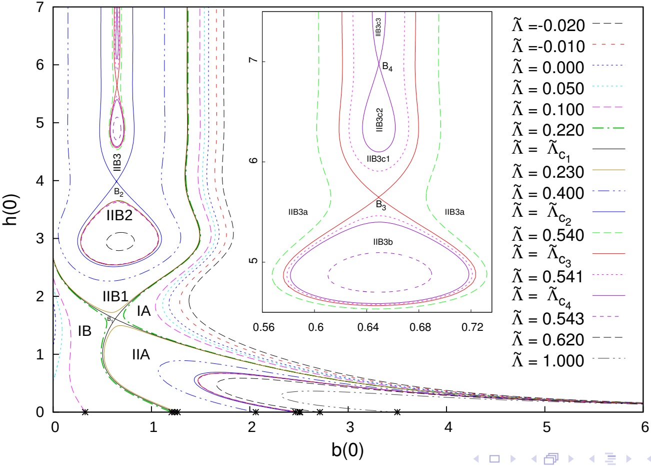
- **In the bifurcation phenomenon, when a sudden change in the behaviour of the phase trajectories occurs as a parameter passes through a critical value then the phase point corresponding to that critical value of the parameter is called a bifurcation point (BP).**

- Further, if a system contains more than one parameter the each parameter has its with its own bifurcation points.
- **We find a whole series of bifurcation points in phase diagrams of BSs. We have determined up to 4 bifurcation points.**

- **Our work establishes that an infinite sequence of BP's exists in the PD of boson stars!**

Results of Model-A: PRD (Rapid) (2016), PRD (2016)

Figure: **2D-Phase Diagram for $(b(0), h(0))$ for different values of $\tilde{\Lambda}$. Asterisks represent transition points from boson stars to boson shells.**



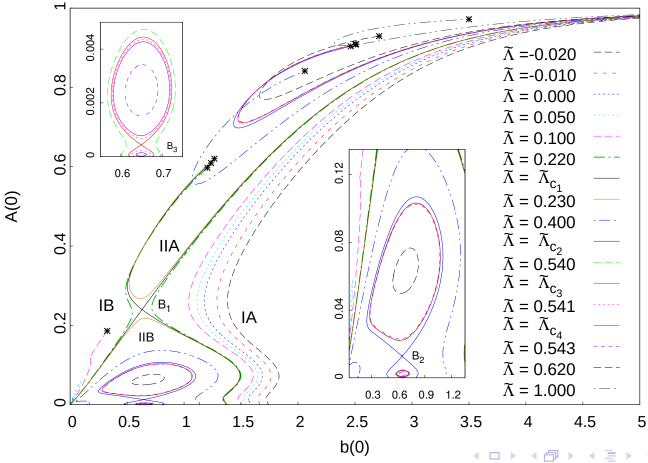
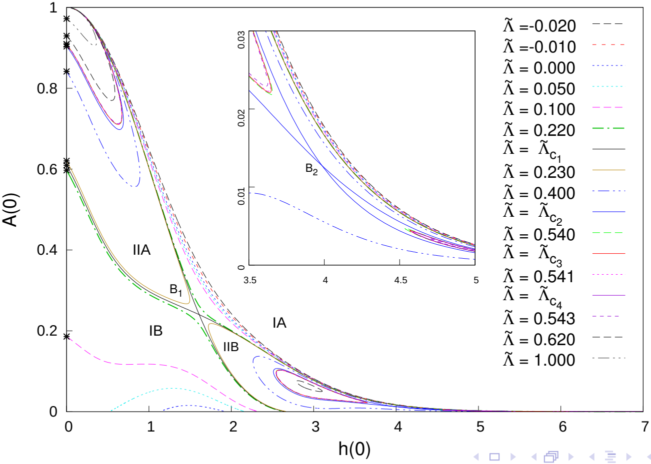


Figure: **2D-Phase Diagram for ($b(0), A(0)$) for different values of $\tilde{\Lambda}$. (The asterisks represent transition points from boson stars to boson shells).**

Figure: **2D-Phase Diagram for $(h(0), A(0))$ for different values of $\tilde{\Lambda}$. (The asterisks represent transition points from boson stars to boson shells).**



3D Plots for Model-A: PRD (2016), Viewing Angle:

- Our 3D-plots of $(h(0), b(0), A(0))$ and $(h(0), b(0), \hat{r}_0)$ with different viewing angles for several values of CConst. $\tilde{\Lambda}$ (where, $\hat{r}_0$ denotes the radius of the boson star).
- **For viewing angles denoted by (θ, ϕ) we use the convention:**
- **Here, θ denotes angle of rotation about the ox-axis in anticlockwise direction and it can take values between 0 to π.**
- **and ϕ is the angle of rotation along the oz'-axis also in anticlockwise direction and it can take values between 0 to 2π.**
- In all our PD's, the astericks denote the transition points from boson stars to boson shells and the insets in the Figs. magnify the parts of the diagram.

Figure: **3D-Phase Diagram for $(A(0), b(0), h(0))$ for different values of $\tilde{\Lambda}$ for viewing angle $(\theta, \phi) \equiv (60, 170)$.**

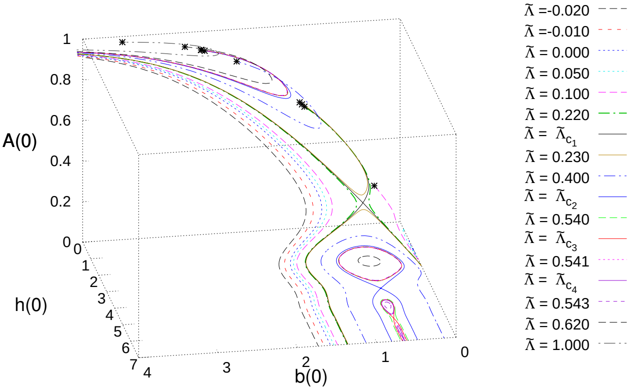


Figure: **3D-Phase Diagram for $A(0), b(0), h(0)$ for different values of $\tilde{\Lambda}$ for viewing angle $(\theta, \phi) \equiv (60, 240)$.**

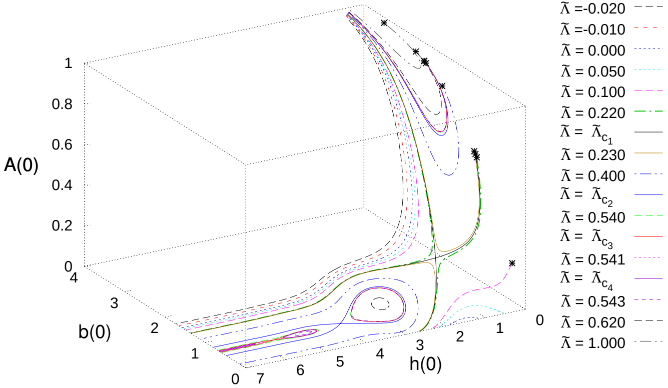
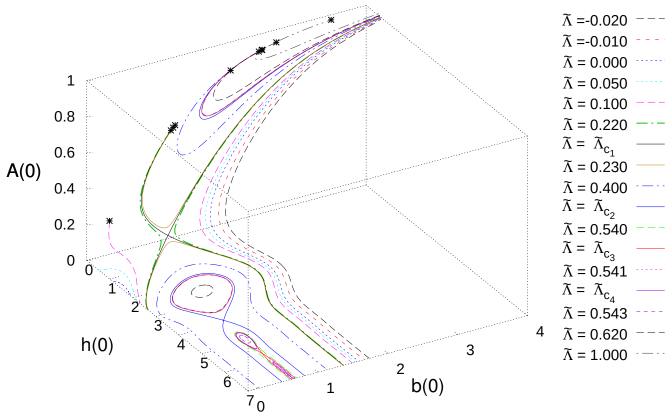
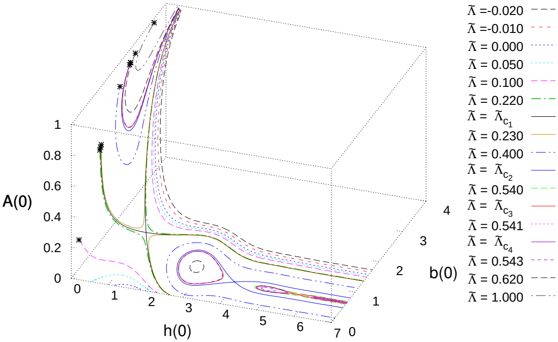


Figure: **3D-Phase Diagram for $A(0), b(0), h(0)$ for different values of $\tilde{\Lambda}$ for viewing angle $(\theta, \phi) \equiv (130, 30)$.**



Model-A: Viewing Angle: $(\theta, \phi) \equiv (130, 70)$ PRD (2016)

Figure: **3D-Phase Diagram for $A(0), b(0), h(0)$ for different values of $\tilde{\Lambda}$ for viewing angle $(\theta, \phi) \equiv (130, 70)$.**



(a)

Model-A: Viewing Angle: $(\theta, \phi) \equiv (130, 350)$ PRD (2016)

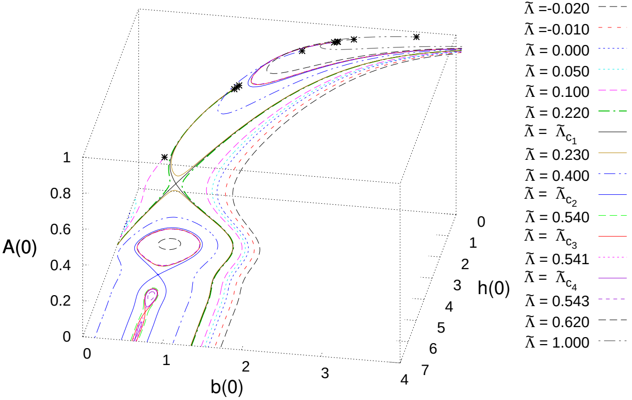
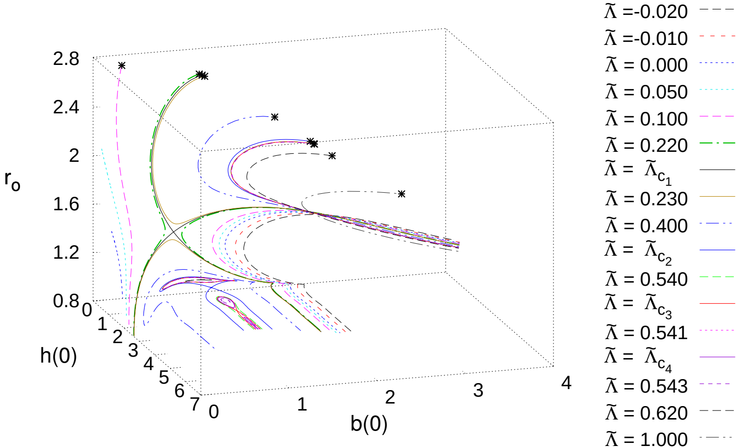


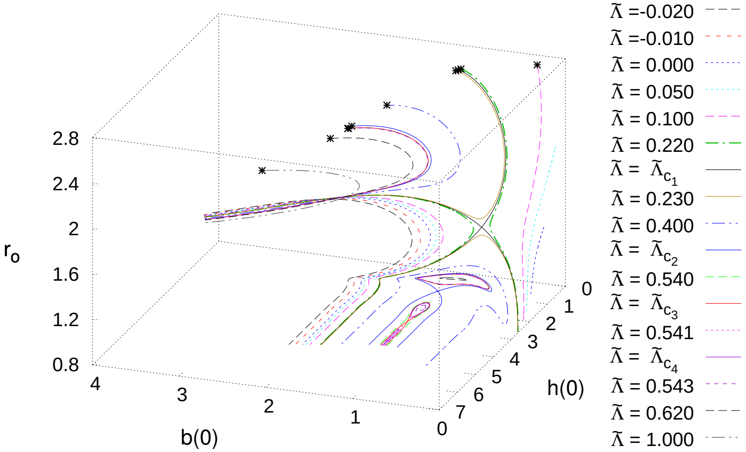
Figure: **3D-Phase Diagram for $A(0), b(0), h(0)$ for different values of $\tilde{\Lambda}$ for viewing angle $(\theta, \phi) \equiv (130, 350)$.**

Model-A: Viewing Angle: $(\theta, \phi) \equiv (112, 17)$ PRD (2016)



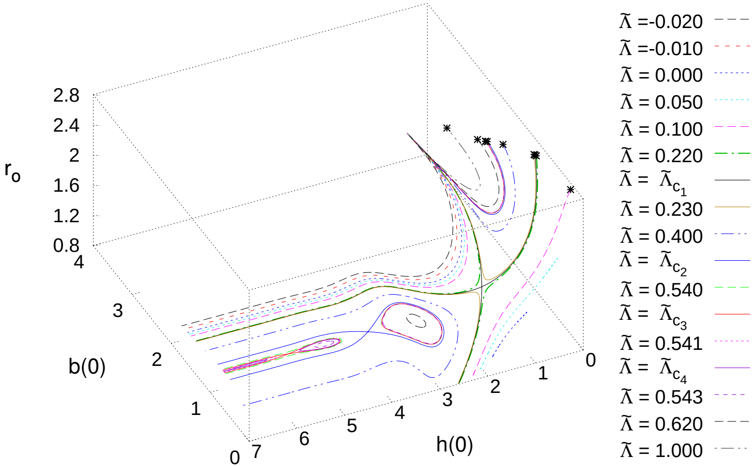
(b)

Model-A: Viewing Angle: $(\theta, \phi) \equiv (60, 200)$ PRD (2016)



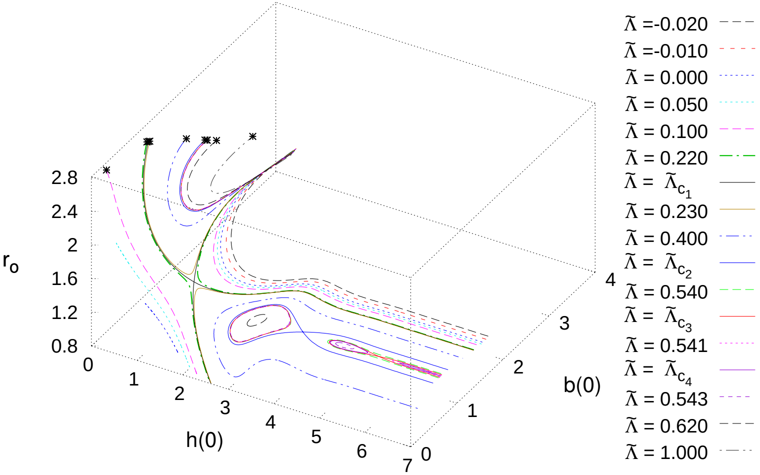
(c)

Model-A: Viewing Angle: $(\theta, \phi) \equiv (35, 245)$ PRD (2016)



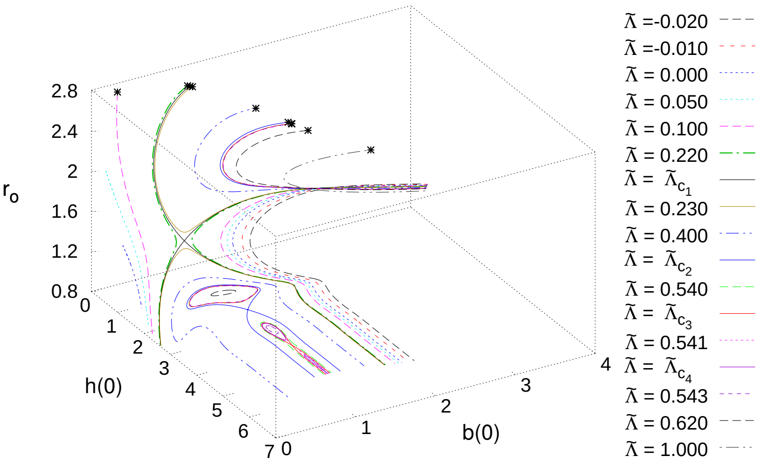
(d)

Model-A: Viewing Angle: $(\theta, \phi) \equiv (140, 60)$ PRD (2016)



(e)

Model-A: Viewing Angle: $(\theta, \phi) \equiv (130, 30)$ PRD (2016)



(f)

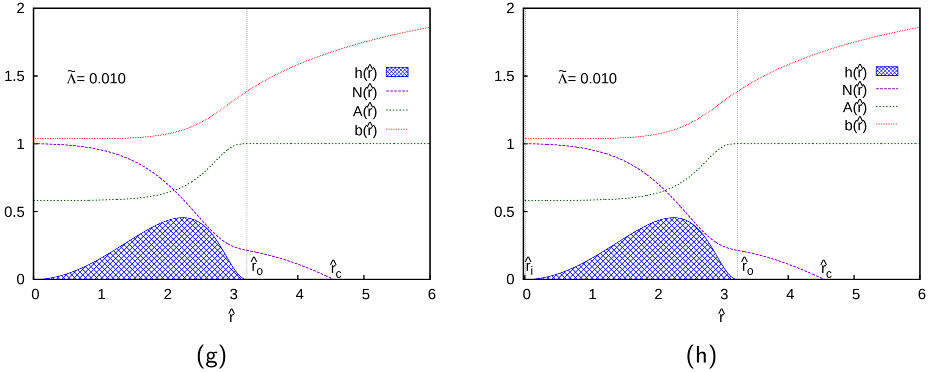
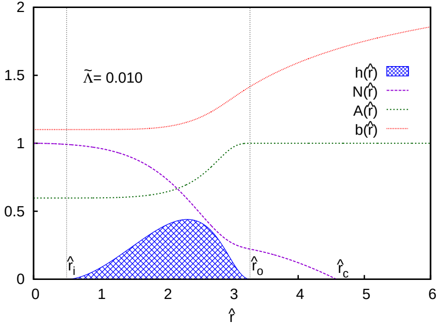
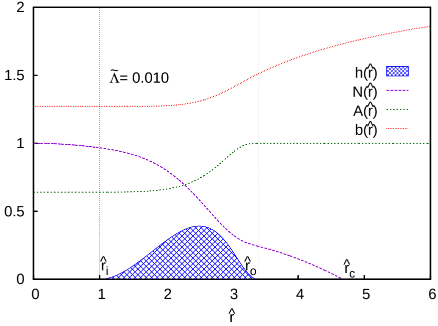


Figure: These figures depict plots of $h(\hat{r})$, $N(\hat{r})$, $A(\hat{r})$ and $b(\hat{r})$ versus $\hat{r}$ for $\tilde{\Lambda} = 0.010$ (which corresponds to the dS space). They correspond to transition points from bosons stars to boson shells (– When we increase value of r_i from zero, Boson Star starts becoming Boson Shell)

Distribution of Fields in dS BShells: Model-A PRD(2016)



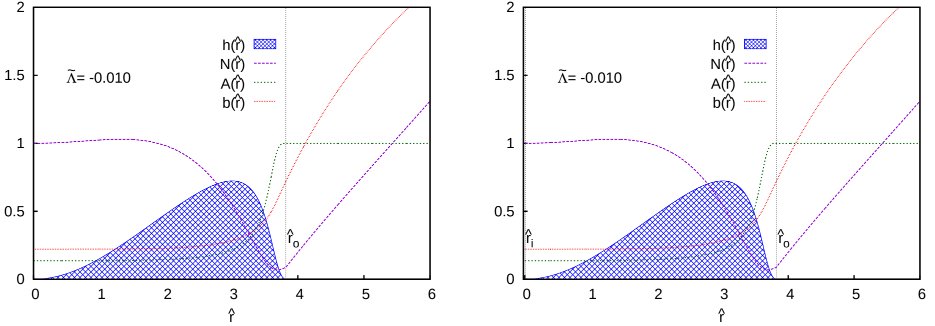
(a)



(b)

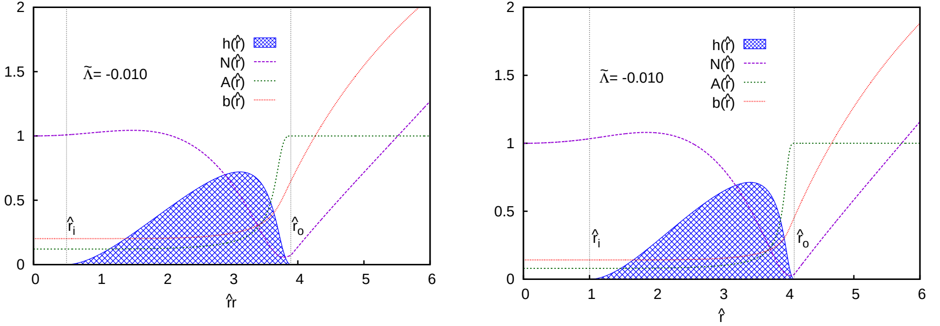
Figure: These figures depict plots of $h(\hat{r})$, $N(\hat{r})$, $A(\hat{r})$ and $b(\hat{r})$ versus $\hat{r}$ for $\tilde{\Lambda} = 0.010$ (which corresponds to the dS space).

Distribution of Fields in AdS BStars: Model-A PRD(2016)



(a) (b)

Figure: These figures depict plots of $h(\hat{r})$, $N(\hat{r})$, $A(\hat{r})$ and $b(\hat{r})$ versus $\hat{r}$ for $\tilde{\Lambda} = -0.010$ (which corresponds to the AdS space).



(a) (b)

Figure: These Figs. depict plots of $h(\hat{r})$, $N(\hat{r})$, $A(\hat{r})$ and $b(\hat{r})$ versus $\hat{r}$ for $\tilde{\Lambda} = -0.010$ (which corresponds to the AdS space).

Model-B: Set $m = 0$ and $\Lambda = 0$ in Model-A

- **Model-B is obtained by setting $m = 0$ and $\Lambda = 0$ in Model-A.**
- **Published in PLB(2017), FBS(2018), EPJC(2019) (Collaboration: Sanjeev K, Usha K, Jutta K, Sarah K, DSK)**

$$
\begin{aligned}
S &= \int \left[\frac{R}{16\pi G} + \mathcal{L}_M \right] \sqrt{-g}\; d^4 x \,, \\
\mathcal{L}_M &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - (D_\mu \Phi)^* (D^\mu \Phi) - V(|\Phi|) \,, \\
D_\mu \Phi &= (\partial_\mu \Phi + ieA_\mu \Phi) \,, F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) \,, \\
V(|\Phi|) &:= \lambda |\Phi|
\end{aligned}
$$

- **Construction of dim-less Const. parameters and dim-less Field variables with dim-less arguments needs different re-scalings.**

Re-Scalings:

- We introduce Dim-less Const. parameters and Dim-less Field variables:

$$\beta = \frac{\lambda\, e}{\sqrt{2}} \quad , \quad \alpha^2(\; := a) = \frac{4\pi G\, \beta^{2/3}}{e^2}$$

$$h(r) = \frac{(\sqrt{2}\; e\, \phi(r))}{\beta^{1/3}} \quad , \quad b(r) = \frac{(\omega + eA_t(r))}{\beta^{1/3}}$$

- Dim-less coordinate $\hat{r}$ is defined as:

$$\hat{r} := \beta^{1/3}\, r \quad \Longrightarrow \quad \frac{d}{dr} = \beta^{1/3}\frac{d}{d\hat{r}}$$

- This gives Dim-less Field variables with Dim-less arguments $\hat{r}$:

$$h(\hat{r}) = \frac{(\sqrt{2}\; e\, \phi(\hat{r}))}{\beta^{1/3}} \quad , \quad b(\hat{r}) = \frac{(\omega + eA_t(\hat{r}))}{\beta^{1/3}}.$$

- Now Primes will denote derivatives w.r.to Dim-less radial coordinate $\hat{r}$

Final set of EOM to be solved numerically:

- Final set of EOM to be solved numerically:

$$
\begin{aligned}
h'' &= \left[\frac{\alpha^2\,\hat{r}h'}{A^2N}\left(2A^2h+\,b'^2\right)-\frac{h'\left(1+N\right)}{\hat{r}N}\right.\\
&\qquad\left.+\frac{A^2N\,\mathrm{sign}(h)-b^2h}{A^2N^2}\right]\\
b'' &= \left[\frac{\alpha^2}{A^2N^2}\hat{r}b'\left(A^2N^2h'^2+b^2h^2\right)-\frac{2b'}{\hat{r}}+\frac{bh^2}{N}\right]\\
N' &= \left[\frac{1-N}{\hat{r}}-\frac{\alpha^2\hat{r}}{A^2N}\left(A^2N^2h'^2+Nb'^2+b^2h^2+2A^2Nh\right)\right]\\
A' &= \left[\frac{\alpha^2\hat{r}}{AN^2}\left(A^2N^2h'^2+b^2h^2\right)\right]
\end{aligned}
$$

Mass and Conserved Charge:

- U(1) invariance of theory leads to a conserved Noether current density:

$$j^\mu = -i\, e\, [\Phi (D^\mu \Phi)^* - \Phi^* (D^\mu \Phi)] \ , \quad j^\mu_{;\mu} = 0$$

- whose time component corresponds to the charge density. The global charge Q of the boson star is then given by

$$Q = -\frac{1}{4\pi} \int_0^{\hat{r}_o} j^t \sqrt{-g}\, dr\, d\theta\, d\phi\, , \; j^t = -\frac{h^2(\hat{r}) b(\hat{r})}{A^2(\hat{r}) N(\hat{r})} .$$

- One can read off the Mass from the metric, making use of the fact that the metric outside the compact star corresponds to a Reissner-Nordström metric. Mass M of the boson star solutions is:

$$M = \left(1 - N(\hat{r}_o) + \frac{\alpha^2 Q^2}{\hat{r}_o^2}\right) \frac{\hat{r}_o}{2} .$$

Infinite Sequence of Bifurcation Points:

- Regularity of the occurence of these BP's makes it tempting to conjecture, that perhaps a whole sequence of BP's may be waiting to be discovered (which might display a self-similar pattern).
- Since difference between BP's : $\triangle_n = (a_{c_n} - a_{c_{n-1}})$ decreases strongly, an exponential approach towards a limiting value a_{c_∞} is expected:
- An Ansatz for n-th BP of the type is proposed and found to work:

$$\boxed{a_{c_n} = \left[a_{c_\infty} + f\ exp(-b\ n\ )\right] = \left[\ a_{c_\infty} + f\ a^n\ \right]}$$

- where a_{c_∞}, f , b and $a := exp(-b)$ are Consts.
- **We find that an infinite sequence of BP's exists.**
- Using first 3 BP's, we make a prediction for the 4th BP. We indeed find the 4th BP to be located at $a_{c_4} = 0.16827861$ which agrees with its predicted value.

Results for Model-B: PLB (2017), EPJC (2019)

Figure: **2D-Phase Diagram for $(h(0), b(0))$ for different values of a for Model-B:**

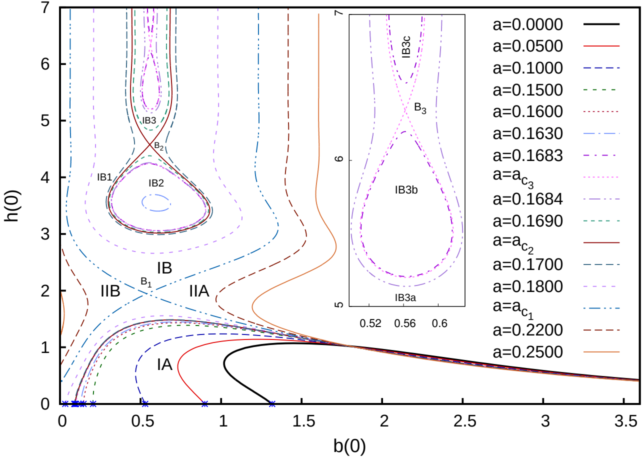


Figure: **2D-Phase Diagram showing 4th BP in the plot ($b(0)$, $h(0)$) for different values of a for Model-B**

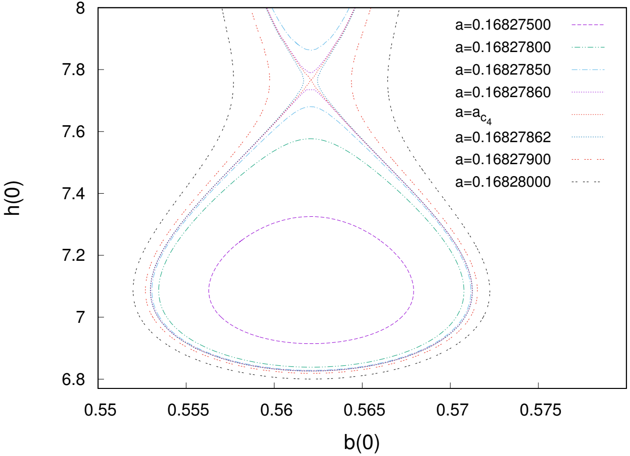
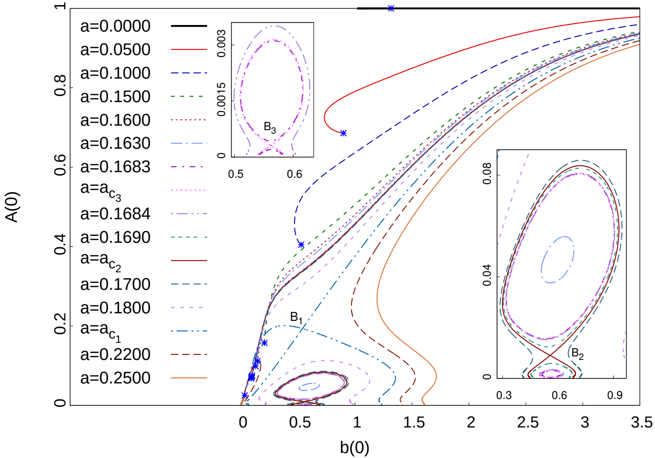


Figure: **2D-Phase Diagram for $(A(0), b(0))$ for different values of a for Model-B**



Results for Model-B: PLB (2017), EPJC (2019)

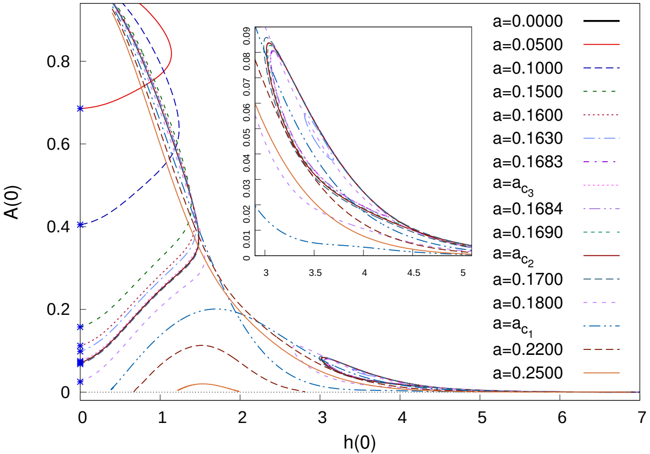
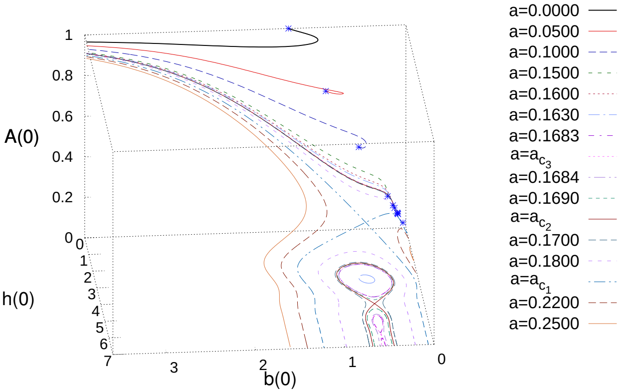


Figure: **2D-Phase Diagram for $(A(0), h(0))$ for different values of a for Model-B**

Results for Model-B: PLB (2017), EPJC (2019)

Figure: **3D-Phase Diagram for $A(0), b(0), h(0)$ for different values of a for viewing angle $(\theta, \phi) \equiv (60, 100)$.**



Results for Model-B: PLB (2017), EPJC (2019)

Figure: **3D-Phase Diagram for $A(0), b(0), h(0)$ for different values of a for viewing angle $(\theta, \phi) \equiv (50, 245)$.**

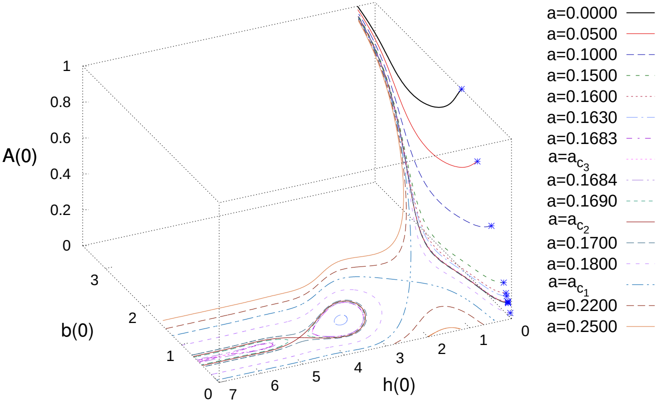
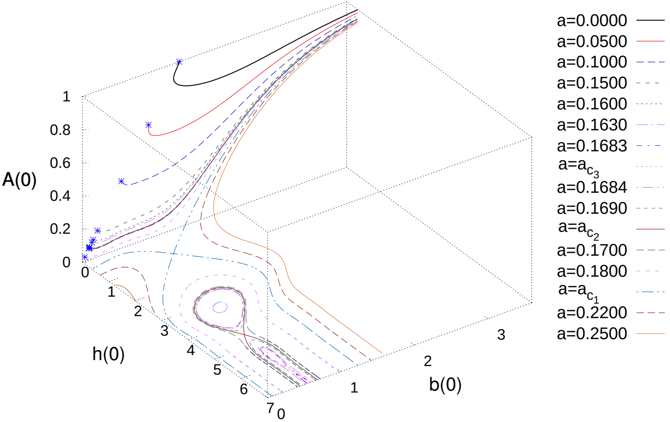


Figure: **3D-Phase Diagram for $A(0), b(0), h(0)$ for different values of a for viewing angle $(\theta, \phi) \equiv (135, 35)$.**



Results for Model-B: PLB (2017), EPJC (2019)

Figure: **3D-Phase Diagram for $A(0), b(0), h(0)$ for different values of a for viewing angle $(\theta, \phi) \equiv (130, 70)$.**

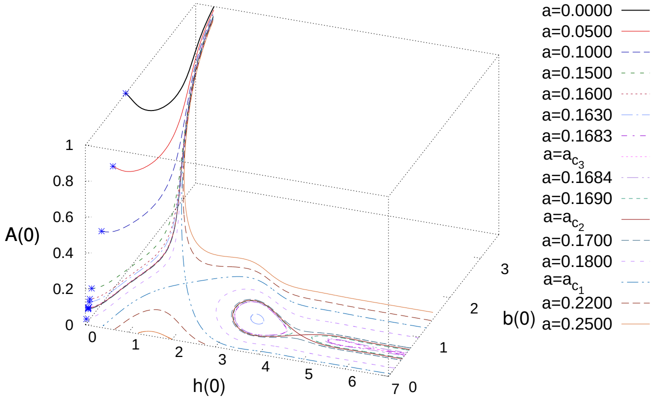


Figure: **3D-Phase Diagram for $A(0), b(0), h(0)$ for different values of a for viewing angle $(\theta, \phi) \equiv (135, 340)$.**

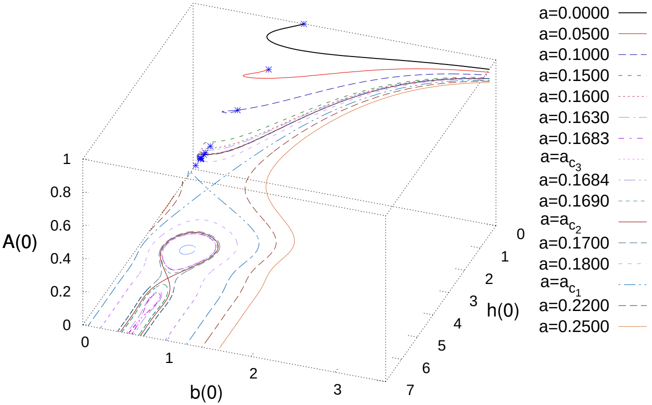
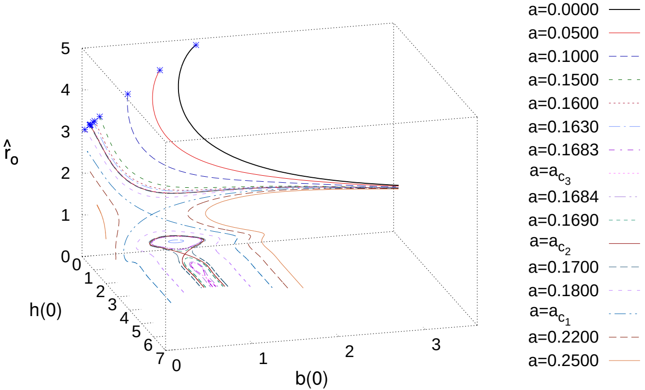
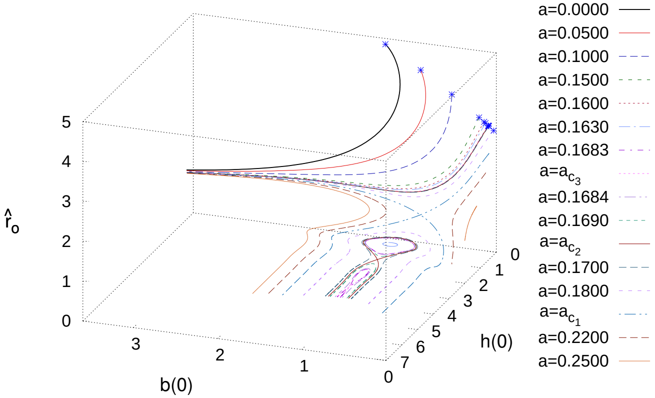


Figure: **3D-Phase Diagram for $h(0), b(0), \hat{r}_o$ for different values of a for viewing angle $(\theta, \phi) \equiv (115, 15)$.**



Results for Model-B: PLB (2017), EPJC (2019)

Figure: **3D-Phase Diagram for $h(0), b(0), \hat{r}_o$ for different values of a for viewing angle $(\theta, \phi) \equiv (60, 200)$.**



Results for Model-B: PLB (2017), EPJC (2019)

Figure: **3D-Phase Diagram for $h(0), b(0), \hat{r}_o$ for different values of a for viewing angle $(\theta, \phi) \equiv (40, 240)$.**

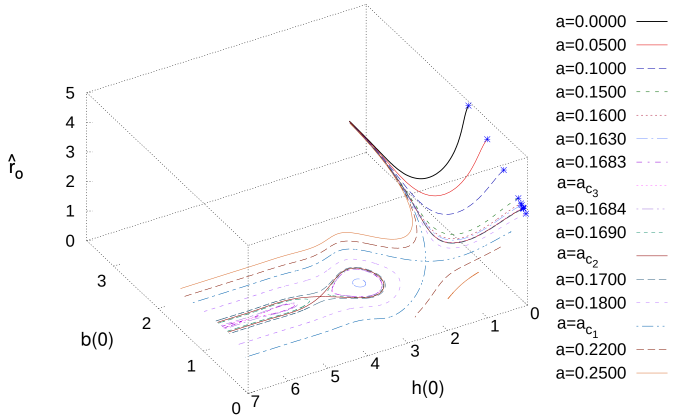
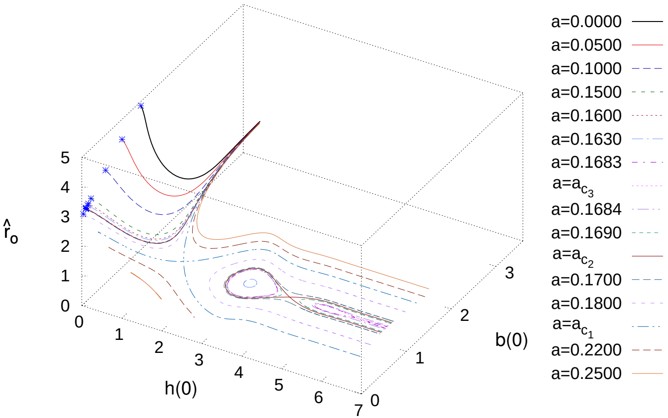
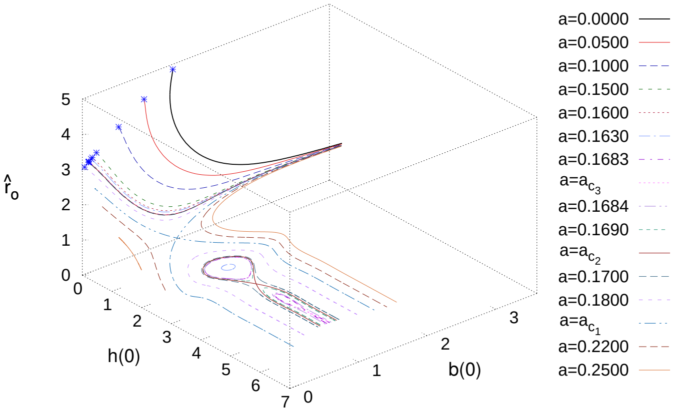


Figure: **3D-Phase Diagram for $h(0), b(0), \hat{r}_o$ for different values of a for viewing angle $(\theta, \phi) \equiv (140, 60)$.**



Results for Model-B: PLB (2017), EPJC (2019)

Figure: **3D-Phase Diagram for $h(0), b(0), \hat{r}_o$ for different values of a for viewing angle $(\theta, \phi) \equiv (130, 40)$.**



- **Action (CQG(2025), GRG(2015))(SK, SA, UK, DSK)):**

$$S = \int \left[\frac{R}{16\pi G} - \xi R|\Phi|^2 - (\nabla_\mu \Phi)(\nabla^\mu \Phi)^* - V(|\Phi|) \right] \sqrt{-g} d^4x$$

$V(|\Phi|) := 2\lambda|\Phi|$; **EoM in Covariant form are:**

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

$$\nabla_\mu \left(\sqrt{-g} \nabla^\mu \Phi \right) = \sqrt{-g}\, \xi\, R\Phi + \lambda \sqrt{-g}\, \frac{\Phi}{|\Phi|}$$

$$\left[\nabla_\mu \left(\sqrt{-g} \nabla^\mu \Phi \right) \right]^* = \sqrt{-g}\, \xi\, R\Phi^* + \lambda \sqrt{-g}\, \frac{\Phi^*}{|\Phi|}$$

$$T_{\mu\nu} = \frac{1}{(1 - 16\pi G\, \xi \Phi^* \Phi)} \Big[2(\nabla_\mu \Phi)^* (\nabla_\nu \Phi) - 2\xi \left(\nabla_\mu \nabla_\nu (\Phi^* \Phi) - \nabla_\alpha \nabla^\alpha (\Phi^* \Phi) g_{\mu\nu} \right) - g_{\mu\nu} \left((\nabla_\alpha \Phi)^* (\nabla_\beta \Phi) \right) g^{\alpha\beta} - 2 g_{\mu\nu} \lambda |\Phi| \Big]$$

Model-C

- **Construct Dim-less Const Parameter α and Dim-less matter field $h(r)$ through:** $\alpha = 8\pi G\,\lambda^2/\omega^4$ **and** $h(r) = \omega^2\phi(r)/\lambda$
- Next we introduce a dimensionless coordinate $\hat{r}$ defined by:

$$\hat{r} := \omega\, r \quad \Longrightarrow \quad \frac{d}{dr} = \omega \frac{d}{d\hat{r}}$$

- **This gives us dim-less matter field $h(\hat{r})$ having dim-less argument $\hat{r}$:** $\qquad h(\hat{r}) = \omega^2\phi(\hat{r})/\lambda$
- The constant parameter ξ is also dimensionless.
- Now onwards, our primes would denote differentiation w.r.to $\hat{r}$
- **We now work with the dim-less constant parameters: α and ξ and dim-less field variables with dim-less arguments:**

- **In Model-C, Final set of Eqs. to be solved numerically is:**

$$N' = \left[\frac{(1-N)(1-2\alpha\xi h^2)}{\hat{r}\big(1-2\alpha\xi(h^2+\hat{r}h\,h'\big)}\right] - \frac{\alpha\hat{r}}{(1-2\alpha\xi(h^2+\hat{r}h\,h'))}\times$$

$$\left[\frac{h^2}{A^2N} + Nh'^2 - 4\xi N\left(h\,h'' + h'^2 + \frac{2h\,h'}{\hat{r}}\right) + 2\,h\right]$$

$$A' = \frac{\alpha\hat{r}\left[h^2 + (1-2\xi)A^2N^2h'^2 - 2\xi A^2N^2h\,h''\right]}{A^2N^2\big(1-2\alpha\xi(h^2+\hat{r}h\,h')\big)}$$

$$h'' = \left(\frac{1-2\alpha\xi(h^2+\hat{r}h\,h')}{1-2\alpha\xi(1-6\xi)h^2}\right)\left[\frac{\mathrm{sign}(h)}{N} - \frac{h}{A^2N^2} - \frac{h'}{\hat{r}}\right.$$

$$\left. + \frac{2\xi h}{\hat{r}^2N} - \frac{2\alpha\xi h\left(h^2-(1-4\xi)A^2N^2h'^2-2A^2Nh\right)}{A^2N^2(1-2\alpha\xi h^2)}\right]$$

$$-\left(h' + \frac{2\xi h}{\hat{r}} + \frac{8\alpha\xi^2h^2h'}{(1-2\alpha\xi h^2)}\right)\times$$

$$\left[\frac{1-2\alpha\xi h^2 + 6\alpha\xi\hat{r}Nh\,h' + 2\alpha\hat{r}\left(\xi\hat{r}Nh'^2-\hat{r}h\right)}{\hat{r}N(1-2\alpha\xi(1-6\xi)h^2)}\right]$$

Model-C: BCs:

- **For obtaining boson star solutions, we fix 5 BCs as:**

$$N(0) = 1\ ,\quad h'(0) = 0\ ,\quad A(\hat{r}_o) = 1\ ,\quad h(\hat{r}_o) = 0 \quad h'(\hat{r}_o) = 0$$

- **This configuration makes the system overdetermined.**
- To address this and solve the differential equations numerically, we introduce the differential equation, $\hat{r}_o' = 0$ in the set of equations, without imposing an additional boundary condition on $\hat{r}_o$.
- This approach allows us to treat $\hat{r}_o$ as a free parameter, effectively balancing the system and enabling a consistent numerical solution.
- **Invariance of theory under an appropriate $U(1)$ phase transformation leads to the conserved Noether current density:**

$$j^\mu = -i\ [\Phi(\nabla^\mu\Phi)^* - \Phi^*(\nabla^\mu\Phi)]\ ,\quad j^\mu{}_{;\mu} = 0$$

Global Charges in Model-C.

- **Its time component corresponds to the charge density ρ:**

$$\rho := j^t = -\frac{\lambda^2\, h^2(\hat{r})}{\omega^3 A^2(\hat{r}) N(\hat{r})}$$

- **The global charge Q of the boson star is then given by:**

$$Q = -\frac{1}{4\pi}\int \rho\sqrt{-g}\, dr\, d\theta\, d\phi = Q_0 \hat{Q}$$

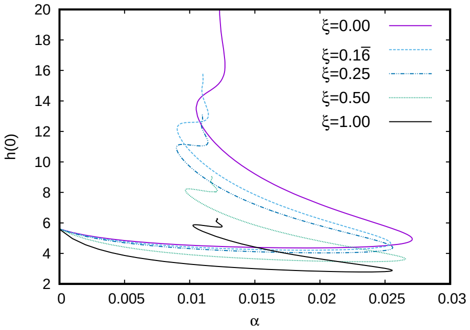
- **where** $Q_0 = \lambda^2/\omega^6$

$$\hat{Q} = \int_0^\infty \frac{h^2(\hat{r})}{A(\hat{r})N(\hat{r})}\hat{r}^2 d\hat{r}$$

- **Mass can be read off the metric, using of the fact that metric outside the star corresponds to a Schwarzschild metric.**
- **Mass parameter of BS solutions:** $\hat{M} = \big(1 - N(\hat{r}_o)\big)\hat{r}_o/2\,.$

Boson star solutions: Scalar field at the center of BS $h(0)$:

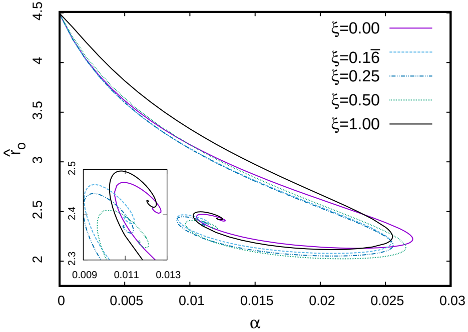
For a fixed value of α, $h(0)$ decreases with increasing ξ. This happens due to a stronger coupling of scalar field with gravity for increasing values of ξ. For $\xi > 0$, beyond $\alpha \to \alpha_{lim}$, $h(0)$ increases monotonically and then exhibits bigger damping effect (as compared to the $\xi = 0$).



(a)

Boson star solutions: The boson star radius $\hat{r}_o$.

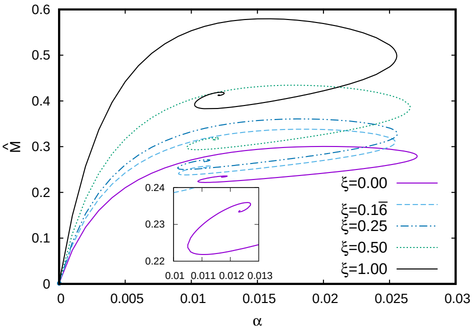
Starting from $\alpha = 0$, $\hat{r}_o$ decreases monotonically along the first branch until $\alpha \to \alpha_{\text{max}}$. Oscillating pattern in the behavior of $h(0)$ for higher branches is reflected as a spiralling pattern of the radius $\hat{r}_o$.
(- increasing ξ indicates a stronger gravity in a theory with NMC).



(b)

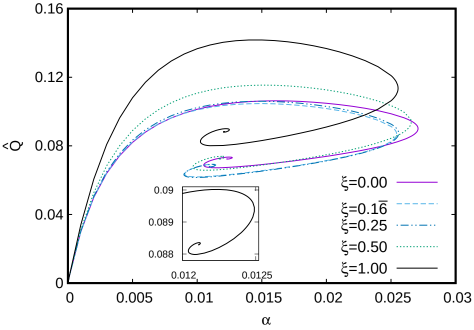
Boson Star Solutions: Mass parameter $\hat{M}$.

Larger values of ξ imply stronger coupling of scalar field with gravity.
This allows the star to sustain higher masses and enable the formation of more massive and compact boson stars.
(- increasing values of ξ lead to larger masses and more compact stars).



(c)

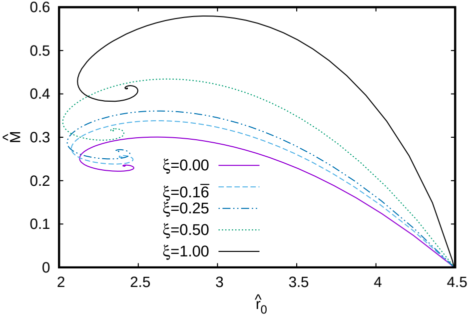
Variation of $\hat{Q}$ with α is similar to that of the Mass parameter $\hat{M}$.
Both $\hat{M}$ and $\hat{Q}$ increase with the increase in α and upon reaching their maximum values, the solutions exhibit familiar spiralling patterns characteristic of highly compact stars.



(d)

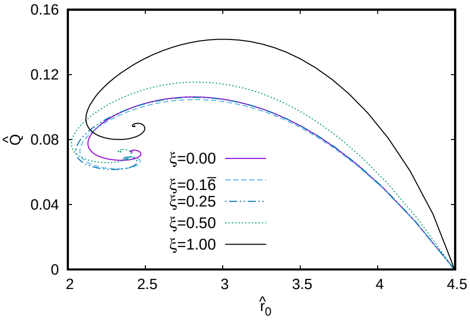
Boson Star Solutions: Mass parameter $\hat{M}$ versus $\hat{r}_o$.

Mass parameter converges towards a limiting values at the center of the spirals, where $\alpha \to \alpha_{lim}$. We depict the Mass $\hat{M}$ versus the radius $\hat{r}_o$ and we find that $\hat{M}$ increases with decrease in the radius of boson star $\hat{r}_0$.



(e)

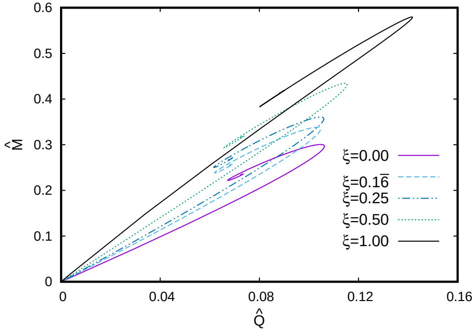
Charge parameter also converges towards a limiting values at the center of the spirals, where $\alpha \to \alpha_{lim}$. We depict the Charge $\hat{Q}$ versus the radius $\hat{r}_o$ and we find that $\hat{Q}$ increases with decrease in the radius of boson star $\hat{r}_0$.



(f)

Plot of Mass parameter $\hat{M}$ versus charge parameter $\hat{Q}$

For $\hat{M}$ vs $\hat{Q}$ and we find that along the first branch of boson star solutions, the mass increases linearly with the charge as α increases from $(\alpha = 0)$.
After $\alpha = \alpha_{max}$, a spiralling behaviour is observed.
Boson stars become more massive as we increase ξ for a fixed value of α.



(g)

Figure: Plot of Mass parameter $\hat{M}$ versus charge parameter $\hat{Q}$ of the boson star

“Stars in the heavens sing a music, if we only had ears to hear” - Pythagorous!

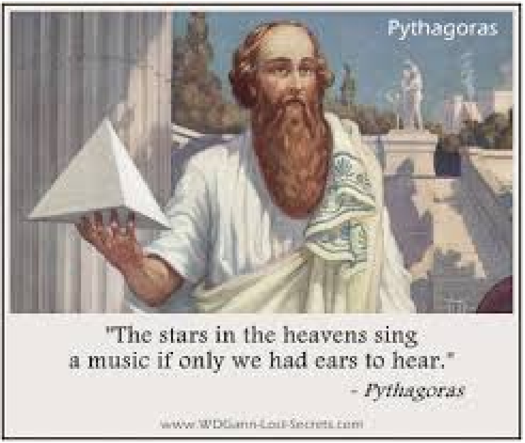


Figure: Pythagorous: Stars in the Heavens Sing a Music